Theorem 2.4.1 (Existence and Uniqueness of Solutions of Linear Equations)

\[ y' + p(t) y = g(t), \quad y(t_0) = y_0 \]

If there is an interval \((a, b)\) to such that \(p(t)\) and \(g(t)\) are continuous in \((a, b)\), then there is a unique solution in \((a, b)\).
**Problem 1.** Determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

(a.) 
\[(4 - t^2)y' + 2ty = 3t^2, \quad y(-3) = 1\]

Dividing by \(4 - t^2\),

\[y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}\]

\[4 - t^2 = 0 \iff 4 = t^2 \iff t = \pm 2\]

(b.)

\[(t - 3)y' + (\ln t)y = 2t, \quad y(1) = 2\]

Dividing by \(t-3\),

\[y' + \frac{\ln t}{t-3} y = \frac{2t}{t-3}\]

\[t - 3 = 0 \iff t = 3\]

Points of discontinuity of \(p\) or \(g\): 
\[\ln t \implies t \leq 0\]
Thm 2.4.2 (Existence and Uniqueness of Solutions of nonlinear equations)

\[ y'(t) = f(t, y), \quad y(t_0) = y_0 \]

If there is a rectangle \((\alpha, \beta) \times (y, s)\) such that \(t \in (\alpha, \beta)\) and \(y_0 \in (y, s)\) and 

\[ f, \ f_y \text{ are continuous}, \]

then there is a unique solution on some subinterval \((\alpha_1, \beta_1)\) of \((\alpha, \beta)\).
Problem 2. State where in the ty-plane the hypotheses of Theorem 2.4.2 (existence and uniqueness theorem for nonlinear equations) are satisfied.

a. \[ y' = (t^2 + y^2)^{3/2} \]
\[ f(t, y) = (t^2 + y^2)^{3/4} \]
\[ f_y(t, y) = \frac{3}{2y} (t^2 + y^2)^{1/2} = \frac{3}{2} (t^2 + y^2)^{1/2} \cdot 2y \]
\[ = 3 (t^2 + y^2)^{1/2} \cdot y \]
\[ \Rightarrow f, f_y \text{ are both continuous everywhere.} \]

b. \[ y' = \frac{1+t^2}{3y-y^2} f(t, y) \]
\[ f = \frac{1+t^2}{3y-y^2} \]
\[ f_y = \frac{2}{3y} \left( (1+t^2)(3y-y^2)^{-1} \right) \]
\[ = \left( 1+t^2 \right) (3y-y^2)^{-2} (3-2y) \]
\[ = \frac{-\left( 1+t^2 \right) (3-2y)}{(3y-y^2)^2} \]

\[ 3y-y^2 = 0 \iff y(3-y) = 0 \]
\[ \iff y = 0 \text{ or } y = 3 \]

In the whole plane \((\mathbb{R}^2, (-\infty, \infty) \times (-\infty, \infty))\), \(f, f_y\) are continuous.

Depending on the initial point, the one with initial point in it will be our rectangle.
Problem 3. Solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value $y_0$.

a. \[ y' + y^3 = 0, \quad y(0) = y_0 \]

Separation of Variables:
\[ y' = -y^3 \]
\[ \Leftrightarrow \int \frac{dy}{y^3} = -\int dt \]
\[ \Leftrightarrow -\frac{1}{2} y^{-2} = -t + C_0 \]
\[ \Leftrightarrow y^{-2} = 2t + C_1 \]
\[ \Leftrightarrow y^2 = \frac{1}{2t + C_1} \]
\[ \Rightarrow y = \pm \frac{1}{\sqrt{2t + \frac{1}{y_0^2}}} \text{, } y \neq 0 \]

Domain:
\[ 2t + \frac{1}{y_0^2} > 0 \]
\[ \Leftrightarrow 2t > -\frac{1}{y_0^2} \]
\[ \Leftrightarrow t > -\frac{1}{2y_0^2} \]

For $y, \neq 0$

b. \[ \frac{dy}{dt} = \frac{t^2}{y(1+y^2)} \]
\[ y(0) = y_0 \]
\[ \Rightarrow y = \frac{u + 1}{1 + u^2} \]
\[ \frac{du}{dt} = 3t + 1 \]
\[ \Rightarrow u = \frac{1}{3} \ln |1 + t^3| + C_0 \]
\[ \Rightarrow y = \pm \sqrt{\frac{1}{3} \ln |1 + t^3| + C_1} \]
\[ y(0) = y_0 : \]
\[ \Rightarrow y_0 = \pm \sqrt{\frac{1}{3} \ln (1) + C_1} \]
\[ \Rightarrow y_0 = \pm \sqrt{C_1} \]
\[ \Rightarrow C_1 = y_0^2 \]
Problem 4.

\[ \frac{dy}{dx} = (y - 4)(y - 2)(y + 1) \]

a. Determine the critical (equilibrium) points.
b. Sketch the graph of \( f(y) \) versus \( y \).
c. Draw the phase line.
d. Classify equilibrium points.
e. Sketch several graphs of solutions in the ty-plane.

\[ 0 = (y-4)(y-2)(y+1) \]
\[ \implies y = 4, 2, -1 \quad \text{equilibrium points} \]
\[ \frac{dy}{dx} = 0 \]

\[ f(y) = (y-4)(y-2)(y+1) \to \infty \text{ as } y \to \infty \]

\[ \text{as } y \to \infty \]

\[ \begin{array}{c}
4 < y < \infty : y = 6 : \frac{dy}{dx} = (6-4)(6-2)(6+1) > 0 \\
2 < y < 4 : y = 3 : \frac{dy}{dx} < 0 \\
-1 < y < 2 : y = 0 : \frac{dy}{dx} > 0 \\
y < -1 : y = -2 : \frac{dy}{dx} < 0
\end{array} \]

Phase line.

4: asymptotically unstable
2: stable
-1: unstable

\[ y \]

\[ \uparrow \]
Problem 5.

\[ \frac{dy}{dt} = (y - 3)^2(y - 1)(y + 2)^2 \]

a. Determine the critical (equilibrium) points.
b. Draw the phase line.
c. Classify equilibrium points.
d. Sketch several graphs of solutions in the ty-plane.

\[ (y-3)^2(y-1)(y+2)=0 \iff y = 3, 1, -2 \]

\[ 3 < y < \infty : y = 10: (10-3)^2(10-1)(10+2)^2 > 0 \]
\[ 1 < y < 3: y = 2: y-1=2-1 > 0 \]
\[ -2 < y < 1: y = 0: 0-1 = -1 < 0 \]
\[ y < -2: y = -3: -3-1 < 0 \]

C. 3: asymptotically semistable
   1: unstable
   -2: semistable.

D. [Diagrams of phase line and solution graphs]
**Problem 6.** Another equation that has been used to model population growth is the Gompertz equation

\[
\frac{dy}{dt} = ry \ln \left( \frac{K}{y} \right)
\]

where \( r \) and \( K \) are positive constants.

a. Sketch the graph of \( f(y) \) versus \( y \), find the critical points, and determine whether each is asymptotically stable or unstable.

b. For \( 0 \leq y \leq K \), determine where the graph of \( y \) versus \( t \) is concave up and where it is concave down.

c. Solve the Gompertz equation subject to the initial condition \( y(0) = y_0 \). Hint: You may wish to let \( u = \ln(y/K) \).

\[\begin{align*}
\text{a. } & ry \ln \left( \frac{K}{y} \right) = 0 \iff y = 0 \text{ or } \ln \left( \frac{K}{y} \right) = 0 \iff y = 0, K. \\
\end{align*}\]

\[\begin{align*}
\frac{dy}{dt} = f(y) = ry \ln \left( \frac{K}{y} \right) \\
\iff \frac{dy}{dt} = 0 \\
\iff \frac{dy}{dt} = 0 \\
\iff y = 0, \frac{K}{y} = 1 \iff y = K.
\end{align*}\]

\[\begin{align*}
\text{b. } & \frac{d^2y}{dt^2} = \frac{d}{dt} \left( ry \ln \left( \frac{K}{y} \right) \right) = \frac{d}{dt} \left( ry \left( \ln K - \ln y \right) \right) \\
& = ry' \left( \ln K - \ln y \right) + ry \left( -\frac{1}{y} \right) y' \\
& = ry' \left( \ln K - \ln y - 1 \right) = 0 \\
\iff & \ y' = 0, \ln K - \ln y - 1 = 0 \iff \ln y = \ln K - 1 \\
\iff & \ y = e^{\ln K - 1} \\
\iff & \ y = e^{\ln K} \cdot e^{-1}
\end{align*}\]
\[ y = e^{\ln k} e^{-1} \]

\[ y = \frac{k}{e} \]

- \((0, \frac{k}{e})\): concave up \(\left(\frac{dy}{dt}\right)\) increases

- \((\frac{k}{e}, k)\): concave down \(\left(\frac{dy}{dt}\right)\) decreases

c. \[ \frac{dy}{dt} = r y \ln \left(\frac{k}{y}\right) \]

\[ \Rightarrow \int \frac{dy}{y \ln \left(\frac{k}{y}\right)} = \int r \, dt \]

\[ u = \ln \left(\frac{k}{y}\right) = \ln k - \ln y \]

\[ du = -\frac{dy}{y} \]