§ 3.5. Method of Undetermined Coefficients

\[ ay'' + by' + cy = g(t) \]

Nonhomogeneous

1. Solve \( ay'' + by' + cy = 0 \) and find the general solution:
   \[ y_h = c_1 y_1 + c_2 y_2 \]

2. Solve \( ay'' + by' + cy = g(t) \) for a particular solution:
   \[ y_p = \ldots \text{(Method of Undetermined Coefficients)} \]

3. \( y = y_h + y_p \) is the general solution of the nonhomogeneous problem.

Method of Undetermined Coefficients

Find a particular solution by guess & plug in
1. \( g(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n \) (polynomial)

\[ y_p = t^s (A_0 t^n + A_1 t^{n-1} + \cdots + A_n) \]

Base form

Ex) \( g(t) = \frac{2t+1}{y_p = At+B} \)

2. \( g(t) = (a_0 t^n + \cdots + a_n) e^{\lambda t} \) (Poly) \cdot (exp)

\[ y_p = t^s (A_0 t^n + A_n) e^{\lambda t} \]

Base form

Ex) \( g(t) = 3 \)

\[ y'' + y' + y = 1 \]

\[ y_p = A \]

\[ y'' + y' + y = 0 \]

\( y_p = A \Rightarrow y_p' = y''_p = 0 \)

\( y_0 + 0 + A = 1 \)

\( y_p = 1 \)

3. \( g(t) = (a_0 t^n + \cdots + a_n) e^{\lambda t} \cos(\beta t) \) (or \( \sin(\beta t) \))

\( (\text{poly}) \cdot (\text{exp}) \cdot (\cos \text{ or } \sin) \)

\[ y_p = t^s ((A_0 t^n + A_n) e^{\lambda t} \cos(\beta t) + (B_0 t^n + B_n) e^{\lambda t} \sin(\beta t)) \]

Base form

Ex)
\[ y'' + 4y' + 3y = 0 \]
\[ e^{-3t} \]

\( y_p = e^{-3t} \)

\( y_p = Ae^{3t} \)

\[ y' = -3Ae^{3t} \]

\[ y'' = 9Ae^{3t} \]

\[ 9Ae^{3t} + 4(-3Ae^{3t}) + 3Ae^{3t} = e^{3t} \]

\( 0 = e^{3t} \)

Cannot find a particular solution

\[ y'' + 4y' + 3y = 0 \]

\[ r^2 + 4r + 3 = 0 \]

\[ y_h = C_1 e^{3t} + C_2 e^{-t} \]
\[ r^2 + 4r + 3 = 0 \]
\[ (r+3)(r+1) = 0 \]
\[ r = -3, -1 \]

\[ y_h = c_1 e^{-3t} + c_2 e^{-t} \]

⇒ Your initial guess (base form) wouldn't work if your guess includes a homogeneous solution. Then, we can modify our guess by multiplying by powers of t's.
Problem 1. Find the forms of particular solutions for the differential equations.

(a) $y'' - 3y' + 2y = -2t^2 e^{4t}$
$$y_p = (At^2 + Bt + C) e^{4t}$$

(b) $y'' - 3y' + 2y = -2t e^{2t}$
$$y_p = (At^2 + Bt + C) e^{2t}$$

(c) $y'' - 4y' + 4y = -2t e^{4t}$
$$y_p = (At^2 + Bt + C) e^{4t}$$

(d) $y'' - 4y' + 4y = -2t^2 e^{2t}$
$$y_p = (At^2 + Bt + C) e^{2t}$$
(e) \( y'' - 4y' + 4y = 3e^{2t} \cos(2t) \)

\[ y_p = Ae^{2t} \cos(2t) + Be^{2t} \sin(2t) \]

(f) \( y'' - 4y' + 13y = 3e^{2t} \cos(2t) \)

\[ y_p = Ae^{2t} \cos(2t) + Be^{2t} \sin(2t) \]

(g) \( y'' - 4y' + 13y = 3e^{2t} \cos(3t) \)

\[ y_p = (Ae^{2t} \cos(3t) + Be^{2t} \sin(3t))e^{3t} \]

\[ y'' - 4y' + 13y = 0 \]
\[ r^2 - 4r + 13 = 0 \]
\[ r = \frac{4 \pm \sqrt{16 - 52}}{2} \]
\[ = 2 \pm 3i \]

\[ y_h = C_1 e^{2t} \cos(3t) + C_2 e^{2t} \sin(3t) \]
Problem 2. Find the general solutions of the given differential equations using the method of undetermined coefficients.

(a) \[ y'' - 2y' - 3y = 3e^{2t} \]

1. \[ y'' - 2y' - 3y = 0 \]
   \[ r^2 - 2r - 3 = 0 \]
   \[ (r-3)(r+1) = 0 \]
   \[ r = 3, -1 \]
   \[ y_h = C_1 e^{3t} + C_2 e^{-t} \]

2. \[ y'' - 2y' - 3y = 3e^{2t} \]

   Guess \[ y_p = Ae^{2t} \] and plug in:
   \[ y_p = Ae^{2t} \]
   \[ y_p' = 2Ae^{2t} \]
   \[ y_p'' = 4Ae^{2t} \]
   \[ \Rightarrow (4Ae^{2t}) - 2(2Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t} \]
   \[ \Rightarrow 4A - 4A - 3A = 3 \]
   \[ \Rightarrow A = -1 \]
   \[ y_p = -e^{2t} \]

3. The general solution
   \[ y = y_h + y_p = C_1 e^{3t} + C_2 e^{-t} - e^{2t} \]
(b)

1. \( y_h = C_1 e^{3t} + C_2 e^t \)
2. \( y_p = (At + B) e^t - t 
= (At^2 + Bt)e^{-t} \)

\[ y_p' = (2At + B)e^{-t} + (At^2 + Bt)(-e^{-t}) \]
\[ = (-A + (2A-B)t + B)e^{-t} \]

\[ y_p'' = (2At + (2A-B))e^{-t} + (-A + (2A-B)t + B)(-e^{-t}) \]
\[ = (A + (-4A + B)t + (2A - 2B))e^{-t} \]

Coeff for \( t e^t \): \( A = -2, (A) = 2, (A) = 0 \) \( \implies A + 2A - 3A = 0 \) \( \implies 0 = 0 \)

Coeff for \( e^t \): \( -2(2A - B) - 3(B) = -3 \) \( \implies -8A = -3 \) \( \implies A = \frac{3}{8} \)

Coeff for \( t e^{-t} \): \( 2A - 4B = 0 \) \( \implies B = \frac{1}{2} A = \frac{3}{16} \)

So,
\[ y_p = \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t} \] and

the general solution
\[ y = y_h + y_p = C_1 e^{3t} + C_2 e^t + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t} . \]