Math 308 (WIR #2).

Prob 1. $y' + \frac{2}{t} y = \frac{\cos(t)}{t^2}$, $y(\pi) = 0$, $(t > 0)$.

Solution. $P(t) = \frac{2}{t}$ and $f(t) = \frac{\cos(t)}{t^2}$.

Integrating factor: $\mu(t) = e^{\int P(t)dt}$

$$\Rightarrow \mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2.$$

Then $$y = \frac{\int \mu(t) f(t)\, dt}{\mu(t)} = \frac{\int t^2 \cdot \frac{\cos(t)}{t^2} dt}{t^2}$$

$$= \frac{\sin(t) + C}{t^2}$$

By $y(\pi) = 0 \Rightarrow 0 = \frac{\sin(\pi) + C}{\pi^2} \Rightarrow C = 0.$

So $y = \frac{\sin(t)}{t^2}$, $(t > 0)$

Prob 2. $y' + \frac{1}{4} y = 3 + 2\cos(2t) e^{-\frac{t}{4}}$, $y(0) = 0$.

Solution. $P(t) = \frac{1}{4}$ and $f(t) = 3 + 2\cos(2t) e^{-\frac{t}{4}}$

Integrating factor:

$$\mu(t) = e^{\int P(t)dt} = e^{\int \frac{1}{4} dt} = e^{\frac{1}{4}t}$$

Then $y = \dfrac{\int \mu(t) f(t)\, dt}{\mu(t)} = \dfrac{\int e^{\frac{1}{4}t} \left(3 + 2\cos(2t) e^{-\frac{t}{4}} \right) dt}{e^{\frac{1}{4}t}}$

$$= \frac{\int \left[3e^{\frac{1}{4}t} + 2\cos(2t) \right] dt}{e^{\frac{1}{4}t}}$$

$$= \frac{12e^{\frac{1}{4}t} + \sin(2t) + C}{e^{\frac{1}{4}t}} = 12 + e^{-\frac{1}{4}t}\sin(2t) + Ce^{-\frac{1}{4}t}$$

By $y(0) = 0$: $0 = 12 + C \Rightarrow C = -12$

So $y = 12 + e^{-\frac{1}{4}t}\sin(2t) - 12e^{-\frac{1}{4}t}$.

As $t \to \infty$, $\lim_{y \to \infty} y = 12$ So $y \approx 12$ (as $t \to \infty$)

Prob. 3 $(x^2 - 9)\dfrac{dy}{dx} + xy = 0$

(a) $x > 3$ (b) $y(0) = 1$.

<u>Solution</u> Assume $x^2 - 9 \neq 0$. $\Rightarrow$ separate into 3 intervals.

$(-\infty, -3)$, or $(-3, 3)$, or $(3, \infty)$.

(a) By $x > 3$: pick $(3, \infty)$ as interval of validity.

Then $y' + \dfrac{x}{x^2 - 9} y = 0$.

$P(x) = \dfrac{x}{x^2 - 9}$ and $f(x) = 0$.

Integrating factor: $\mu(x) = e^{\int P(x)dx} = e^{\int \frac{x}{x^2-9}dx}$

$$= e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2-9} \qquad (x>3)$$

So $$y = \frac{\int \mu(x) f(x)\,dx}{\mu(x)} = \frac{\int 0\,dx}{\mu(x)} = \frac{c}{\sqrt{x^2-9}}$$

on $(3,\infty)$.

(b) By $y(0)=1$ $\begin{cases} x=0 \\ y=1 \end{cases}$ pick $(-3,3)$ as interval of validity.

Then $y' + \frac{x}{x^2-9}y = 0$. $P(x) = \frac{x}{x^2-9}$ and $f(x)=0$.

Integrating factor: $\mu(x) = e^{\int \frac{x}{x^2-9}dx} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{9-x^2} \quad (-3<x<3)$

So $$y = \frac{\int \mu(x) f(x)\,dx}{\mu(x)} = \frac{\int 0\,dx}{\sqrt{9-x^2}} = \frac{c}{\sqrt{9-x^2}} \text{ on } (-3,3).$$

Plug in $\begin{cases} x=0 \\ y=1 \end{cases}$: $1 = \frac{c}{\sqrt{9}} \Rightarrow c=3$.

Then $y = \frac{3}{\sqrt{9-x^2}}$ on $(-3,3)$

Prob 4. $xy' = (1-y^2)^{1/2}$

Solution. Separate variables $x \cdot \frac{dy}{dx} = (1-y^2)^{1/2}$

$(1-y^2)^{1/2} \neq 0$ and $x \neq 0$: $\quad \dfrac{dy}{\sqrt{1-y^2}} = \dfrac{dx}{x}$.

Integrate : $\quad \displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x}$.

$\Rightarrow \quad \arcsin(y) = \ln|x| + C$

Then $\quad y = \sin(\ln|x| + C)$. $\quad (x \neq 0)$

When $(1-y^2)^{1/2} = 0 \quad \Rightarrow \quad y = \pm 1$ (constantly)

They also satisfy the ODE :

$y = \sin(\ln|x| + C)$, or $y = 1$, or $y = -1$.

$(x \neq 0)$

Prob. 5 $\quad y' = \dfrac{3x^2 - e^x}{2y - 5}$, $\quad y(0) = 1$.

Solution. $\quad \dfrac{dy}{dx} = \dfrac{3x^2 - e^x}{2y-5}$ $\quad$ separate variables

$(2y-5)\,dy = (3x^2 - e^x)\,dx$.

Integrate : $\quad \displaystyle\int (2y-5)\,dy = \int (3x^2 - e^x)\,dx$

$\Rightarrow \quad y^2 - 5y = x^3 - e^x + C$

quadratic formula: $\quad y = \dfrac{5 \pm \sqrt{25 + 4(x^3 - e^x + C)}}{2}$.

Use $y(0) = 1$: $\quad 1 = \dfrac{5 \pm \sqrt{25 + 4(-1+C)}}{2}$

$$\Rightarrow \quad -3 = \pm\sqrt{25+4(C-1)} \quad (\text{choose "$-$" sign}).$$

$$\Rightarrow \quad 3 = \sqrt{25+4C-4} \quad \Rightarrow \quad C = -3$$

Then
$$y = \frac{5 - \sqrt{4x^3 - 4e^x + 13}}{2}$$

Prob 6. $\sin(2x)\,dx + \cos(3y)\,dy = 0$, $y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$.

Solution. separate: $\cos(3y)\,dy = -\sin(2x)\,dx$

integrate: $\int \cos(3y)\,dy = \int -\sin(2x)\,dx$

$$\Rightarrow \quad \frac{1}{3}\sin(3y) = \frac{1}{2}\cos(2x) + C$$

or $\quad \sin(3y) = \frac{3}{2}\cos(2x) + C$.

Plug in $y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$: $\begin{cases} x = \frac{\pi}{2} \\ y = \frac{\pi}{3} \end{cases}$

$$\Rightarrow \quad \sin(\pi) = \frac{3}{2}\cos(\pi) + C$$

$$\Rightarrow \quad 0 = -\frac{3}{2} + C \quad \text{or} \quad C = \frac{3}{2}.$$

Then $\quad \sin(3y) = \frac{3}{2}\cos(2x) + \frac{3}{2}$

Note: arcsin ranges from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

and since $y\left(\frac{\pi}{2}\right) = \frac{\pi}{3}$ ($3y$ attains π)

Then $3y = \pi - \arcsin\left(\frac{3}{2}\cos(2x) + \frac{3}{2}\right).$

Hence. $y = \frac{\pi}{3} - \frac{1}{3}\arcsin\left(\frac{3}{2}\cos(2x) + \frac{3}{2}\right).$

Prob. 7. Let y be the amount of dye after t mins.

Then $y(0) = 200 \times 1 = 200$ g.

Set up an ODE based on rate of change:

$$y' = (\text{rate in}) - (\text{rate out})$$
$$= 0 - \frac{y}{200} \times 2$$
$$= -\frac{y}{100}$$

Separate variables: $\frac{dy}{y} = -\frac{1}{100}\,dt$ $(y \neq 0).$

$$\int \frac{dy}{y} = \int -\frac{1}{100}\,dt$$

$\Rightarrow$ $\ln y = -\frac{1}{100}t + C.$ $(y>0).$

$$y = Ce^{-\frac{1}{100}t}.$$

Use $y(0) = 200$: $C = 200$

So $y = 200e^{-\frac{t}{100}}.$

Answer the question: put $1\% \cdot y(0) = 200e^{-\frac{t}{100}}$

So $2 = 200\, e^{-\frac{t}{100}}$ (Solve for t).

$\Rightarrow$ $e^{\frac{t}{100}} = 100$

$\Rightarrow$ $\frac{t}{100} = \ln 100$

Then $t = 100 \ln 100$

Prob. 8. Let $y(t)$ be the temperature of the coffee.

Then $y(0) = 200$ F and $y(1) = 190$ F.

Set up an ODE based on rate of change:

$y' = k(y - 70)$ for some constant k.

Separate variables: $\frac{dy}{y-70} = k\,dt$. $(y > 70)$ greater than room temp.

integrate: $\int \frac{dy}{y-70} = \int k\,dt$

$\Rightarrow$ $\ln(y-70) = kt + C$

$\Rightarrow$ $y - 70 = Ce^{kt}$

$\Rightarrow$ $y = 70 + Ce^{kt}$.

Use $y(0)=200$: $200 = 70 + C \Rightarrow C = 130.$

Use $y(1)=190$: $190 = 70 + 130e^{k} \Rightarrow k = \ln \frac{12}{13}$

So $y = 70 + 130 e^{t \ln \frac{12}{13}}$

Answer the question : put $150 = 70 + 130 e^{t \ln \frac{12}{13}}$

So $130 e^{t \ln \frac{12}{13}} = 80$ (solve for t).

$\Rightarrow \quad e^{t \ln \frac{12}{13}} = \frac{8}{13}$

$\Rightarrow \quad t \cdot \ln \frac{12}{13} = \ln \frac{8}{13}$

$\Rightarrow \quad t = \dfrac{\ln(8/13)}{\ln(12/13)} \approx 6.066$ mins