Math 308 (WIR #4)

ODE : $(3xy+y^2)dx + (x^2+xy)dy = 0$.

Prob. 1.

Solution $\begin{cases} M = 3xy+y^2 \\ N = x^2+xy \end{cases}$ then $\begin{cases} M_y = 3x+2y \\ N_x = 2x+y \end{cases}$ not exact

Compute $\dfrac{M_y - N_x}{N} = \dfrac{(3x+2y)-(2x+y)}{x^2+xy} = \dfrac{x+y}{x(x+y)} = \dfrac{1}{x}$ only in x

So $k(x) = \frac{1}{x}$ and $\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$.

Multiply μ to ODE :

$$x(3xy+y^2)\,dx + x(x^2+xy)\,dy = 0.$$

Then new $\begin{cases} M = 3x^2y+xy^2 \\ N = x^3+x^2y \end{cases}$ $\begin{cases} M_y = 3x^2+2xy \\ N_x = 3x^2+2xy \end{cases}$ exact ✓

So $F(x,y) = \int (3x^2y+xy^2)\,dx = x^3y + \frac{1}{2}x^2y^2 + g(y)$

Use $F_y = N$: $\frac{\partial}{\partial y}\left(x^3y + \frac{1}{2}x^2y^2 + g(y)\right) = x^3 + x^2y$

$\Rightarrow x^3 + x^2y + g'(y) = x^3 + x^2y$.

$\Rightarrow g'(y) = 0$ and choose $g(y) = 0$.

So $F(x,y) = x^3y + \frac{1}{2}x^2y^2$ and solution curve is

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = C}$$

Prob. 2. Solve $y''+3y'+2y=0$.

Solution. Characteristic eq. $r^2+3r+2=0$.

$\Rightarrow (r+2)(r+1)=0$. $\Rightarrow r_1=-2$ and $r_2=-1$.

Then $y_1=e^{-2t}$ and $y_2=e^{-t}$

General Solution : $\boxed{y=c_1e^{-2t}+c_2e^{-t}}$

Prob. 3. Solve $y''+4y'+3y=0$ with $y(0)=2$ and $y'(0)=-1$.

Solution. Characteristic eq. $r^2+4r+3=0$.

$\Rightarrow (r+3)(r+1)=0 \quad \Rightarrow r_1=-3$ and $r_2=-1$.

Then $y_1=e^{-3t}$ and $y_2=e^{-t}$.

General solution : $y=c_1e^{-3t}+c_2e^{-t}$

Use $y(0)=2$ and $y'(0)=-1$:

Compute $y'=-3c_1e^{-3t}-c_2e^{-t}$

So

$$\begin{cases} y(0)=c_1+c_2=2 \\ y'(0)=-3c_1-c_2=-1 \end{cases} \Rightarrow \begin{cases} c_1=-\frac{1}{2} \\ c_2=\frac{5}{2} \end{cases}$$

Then $\boxed{y=-\frac{1}{2}e^{-3t}+\frac{5}{2}e^{-t}}$.

Prob. 4. Determine interval of validity: $t(t-4)y'' + 3ty' + 4y = 2$.

with $y(3) = 0$ and $y'(3) = -1$.

Solution. Standard form: $y'' + \frac{3t}{t(t-4)} y' + \frac{4}{t(t-4)} y = \frac{2}{t(t-4)}$

So $t \neq 0$ and $t \neq 4$. Then we have 3 candidates

$(-\infty, 0)$, $(0, 4)$, $(4, \infty)$.

It must contain $t_0 = 3$: $\boxed{\text{choose } (0,4)}$

Prob. 5 Find Wronskian of e^{-2t} and te^{-2t}

Solution. $W(y_1, y_2) = \det \begin{pmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{pmatrix}$

$= e^{-2t} \cdot e^{-2t}(1-2t) - te^{-2t} \cdot (-2)e^{-2t}$.

$= e^{-4t}[(1-2t) + 2t] = \boxed{e^{-4t} \neq 0}$

Prob. 6. $y'' + 4y = 0$. Check $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$.

Solution. First check they are solutions:

$y_1' = -2\sin(2t)$ and $y_1'' = -4\cos(2t)$.

$y_1'' + 4y_1 = -4\cos(2t) + 4 \cdot \cos(2t) = 0$ ✓

$y_2' = 2\cos(2t)$ and $y_2'' = -4\sin(2t)$.

$y_2'' + 4y_2 = -4\sin(2t) + 4 \cdot \sin(2t) = 0$ ✓

Second, compute Wronskian and check:

$$W(y_1, y_2) = \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} = 2\cos^2(2t) - (-2)\sin^2(2t)$$

$$= 2 \neq 0.$$

So, yes they form a fundamental set of solutions.

General solution $\boxed{y = c_1\cos(2t) + c_2\sin(2t)}$

Prob. 7. If $W(f,g) = 3e^{4t}$ and $f(t) = e^{2t}$, find $g(t)$.

Solution. By definition:

$$W(f,g) = \det\begin{pmatrix} f & g \\ f' & g' \end{pmatrix} = \det\begin{pmatrix} e^{2t} & g \\ 2e^{2t} & g' \end{pmatrix} = e^{2t}g' - 2e^{2t}g.$$

Then $e^{2t}g' - 2e^{2t}g = 3e^{4t}$

$$\Rightarrow \quad g' - 2g = 3e^{2t}$$

$$\mu(t) = e^{\int(-2)\,dt} = e^{-2t}. \quad \text{and}$$

$$g(t) = \frac{\int \mu(t)\cdot 3e^{2t}\,dt}{\mu(t)} = \frac{\int e^{-2t}\cdot 3e^{2t}\,dt}{e^{-2t}} = \frac{\int 3\,dt}{e^{-2t}}$$

$$= \frac{3t + C}{e^{-2t}} = \boxed{3te^{2t} + Ce^{2t}}$$

Prob. 8. Consider $t^2y'' + ty' - y = 0$. Given $y_1(t) = t$.

Solution. Verify: $y_1' = 1$ and $y_1'' = 0$.

So $t^2\cdot 0 + t\cdot 1 - t = 0$ ✓

Rewrite : (standard form) $y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 0$. Then $P(t) = \frac{1}{t}$

By Abel's formula : $W(t) = Ce^{-\int \frac{1}{t}dt} = Ce^{-\ln t} = Ct^{-1}$.

Simply choose $C = 1$, since we only need one solution.

Then $W(t) = \frac{1}{t}$ and by definition :

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} t & y_2 \\ 1 & y_2' \end{pmatrix}$$

$$= t y_2' - y_2$$

So $\quad t y_2' - y_2 = \frac{1}{t}$ and $y_2' - \frac{1}{t}y_2 = \frac{1}{t^2}$

$$\mu(t) = e^{\int (-\frac{1}{t})dt} = e^{-\ln(t)} = t^{-1} = \frac{1}{t}.$$

and $\quad y_2(t) = \dfrac{\int \frac{1}{t} \cdot \frac{1}{t^2} dt}{\frac{1}{t}} = \dfrac{-\frac{1}{2}t^{-2} + C}{\frac{1}{t}} = -\frac{1}{2} \cdot \frac{1}{t} + Ct$

(Choose $C=0$, since only one solution is needed here).

The general solution is $\quad y = c_1 t + c_2\left(-\frac{1}{2t}\right) = \boxed{c_1 t + \frac{c_2}{t}}$

Prob. 9 $\quad$ Solve $\quad y'' - 2y' + 6y = 0$.

<u>Solution</u>. $\quad$ Characteristic eq. $r^2 - 2r + 6 = 0$.

Quadratic formula : $r = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \cdot 6}}{2}$

$$= 1 \pm \sqrt{5}\, i$$

So $a=1$ and $b=\sqrt{5}$.

$y_1 = e^t\cos(\sqrt{5}t)$ and $y_2 = e^t\sin(\sqrt{5}t)$

General solution : $\boxed{y = c_1 e^t\cos(\sqrt{5}t) + c_2 e^t\sin(\sqrt{5}t)}$

Prob. 10. Solve $y'' - 2y' + 5y = 0$ with $y(\frac{\pi}{2}) = 0$ and $y'(\frac{\pi}{2}) = 2$.

Solution. Characteristic eq. $r^2 - 2r + 5 = 0$.

Quadratic formula : $r = \dfrac{2 \pm \sqrt{(-2)^2 - 4\cdot 5}}{2} = 1 \pm 2i$

So $a=1$ and $b=2$.

$y_1 = e^t\cos(2t)$ and $y_2 = e^t\sin(2t)$

General solution : $y = c_1 e^t\cos(2t) + c_2 e^t\sin(2t)$

Use $y(\frac{\pi}{2}) = 0$ and $y'(\frac{\pi}{2}) = 2$.

Compute $y' = c_1 e^t\cos(2t) - 2c_1 e^t\sin(2t) + c_2 e^t\sin(2t) + 2c_2 e^t\cos(2t)$

Then $\begin{cases} y(\frac{\pi}{2}) = -c_1 e^{\frac{\pi}{2}} = 0. \\ y'(\frac{\pi}{2}) = -c_1 e^{\frac{\pi}{2}} - 2c_2 e^{\frac{\pi}{2}} = 2 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = -e^{-\frac{\pi}{2}} \end{cases}$

Then $\boxed{y = -e^{-\frac{\pi}{2}}\, e^t\sin(2t)}$.