Math 308 (WIR) #5

Problem 1. $y'' - 6y' + 9y = 0$ with $y(0)=0$ and $y'(0)=2$

Solution. Characteristic eq. $r^2 - 6r + 9 = 0$.

$\Rightarrow \quad (r-3)^2 = 0 \quad \Rightarrow \quad r = 3$. (repeated)

Then $y_1 = e^{3t}$ and $y_2 = te^{3t}$.

General solution is $y = c_1 e^{3t} + c_2 t e^{3t}$.

Use $y(0)=0$: $\quad y(0) = c_1 = 0 \quad \Rightarrow \quad c_1 = 0$.

Compute $\quad y' = c_2 e^{3t} + 3c_2 t e^{3t}$

Then $\quad y'(0) = c_2$

Use $y'(0) = 2$: $\quad c_2 = 2$.

So $\quad \boxed{y = 2te^{3t}}$.

Prob. 2. $t^2 y'' + 3ty' + y = 0 \quad (t>0)$. given $y_1(t) = t^{-1}$.

Verify: $y_1' = -t^{-2}$ and $y_1'' = 2t^{-3}$.

So $\quad t^2 y_1'' + 3t y_1' + y_1$

$= t^2(2t^{-3}) + 3t(-t^{-2}) + t^{-1}$

$= 2t^{-1} - 3t^{-1} + t^{-1} = 0 \quad$ ✓

$y_1 = t^{-1}$ is a solution

Let $\quad y_2 = v(t)\, y_1 = v \cdot t^{-1}$.

Compute $y_2' = v' \cdot t^{-1} + v \cdot (-1) t^{-2}$ and $y_2'' = v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}$

Plug back in : $t^2 \left(v'' t^{-1} - 2v' t^{-2} + 2v t^{-3} \right) + 3t \left(v' t^{-1} - v t^{-2} \right) + v t^{-1} = 0$

$\Rightarrow \quad v'' \cdot t - 2v' + 3v' + 2v t^{-1} - 3v t^{-1} + v t^{-1} = 0$

$\Rightarrow \quad v'' t + v' = 0.$

u-sub : Let $u = v'$ then $u' = v''$.

So $u' \cdot t + u = 0$ or $u' + \frac{1}{t} u = 0.$

Integrating factor : $\mu = e^{\int \frac{1}{t} dt} = e^{\ln(t)} = t$

and $u = \dfrac{\int t \cdot 0 \, dt}{t} = \dfrac{C_1}{t}$

So $v' = C_1 t^{-1}$ and $v = C_1 \ln(t) + C_2.$

Simply choose $C_1 = 1$ & $C_2 = 0$. (v is non-constant).

So $\boxed{y_2 = v \cdot y_1 = t^{-1} \ln(t)}$

Prob. 3. $xy' + (2x-3)y = 4x^4$, $y(1) = 0$

<u>Solution</u> Standard form: $y' + \frac{2x-3}{x} y = 4x^3$ $(x \neq 0)$.

So $(-\infty, 0)$ or $(0, \infty)$ choose $(0, \infty)$ containing $x=1$

Integrating factor $\mu(x) = e^{\int \frac{2x-3}{x} dx} = e^{2x - 3\ln x} = e^{2x} \cdot \frac{1}{x^3}$

Then $y = \dfrac{\int e^{2x} \cdot \frac{1}{x^3} \cdot 4x^3 \, dx}{e^{2x} \cdot \frac{1}{x^3}} = \dfrac{2e^{2x} + C}{e^{2x} \cdot \frac{1}{x^3}} = 2x^3 + Cx^3 e^{-2x}.$

Use $y(1)=0$: $\quad y(1) = 2 + Ce^{-2} = 0 \Rightarrow C = -2e^{2}$

So $\boxed{y = 2x^3 - 2e^2 x^3 e^{-2x}}$ on $(0, \infty)$

Prob 4. $\frac{1}{x}\frac{dy}{dx} = y^2 \sin(x)$.

Solution. Separate variables: $\frac{dy}{y^2} = x\sin(x)dx \quad (y^2 \neq 0)$.

$$\Rightarrow \int \frac{dy}{y^2} = \int x \sin(x) dx$$

$$\Rightarrow -\frac{1}{y} = -x\cos(x) - \int (-\cos(x)) \cdot 1 \, dx.$$

$$= -x\cos(x) + \sin(x) + C.$$

So $\boxed{y = \frac{1}{x\cos(x) - \sin(x) + C}}$

When $y^2 = 0 \Rightarrow \boxed{y=0}$ (constant)

Prob 5. $(t^2-1)y' + t^2 y = t-3, \quad y(-2) = 2023$.

Solution. Standard form: $y' + \frac{t^2}{t^2-1} y = \frac{t-3}{t^2-1}$. $\quad (t^2-1 \neq 0)$

So $t \neq \pm 1$ intervals: $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$.

Choose the one containing $t=-2$: $(-\infty, -1)$.

$\frac{t^2}{t^2-1}$ and $\frac{t-3}{t^2-1}$ are continuous on $(-\infty, -1)$.

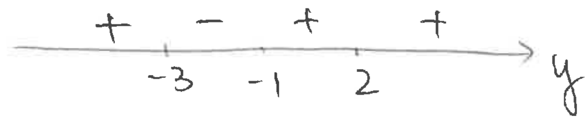
So $(-\infty, -1)$ is the interval.

Prob. 6. $\frac{dy}{dt} = (y-2)^2(y+1)(y+3)$.

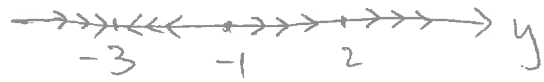
Solution $f(y) = (y-2)^2(y+1)(y+3)$, autonomous.

Equilibrium points: $f(y)=0 \Rightarrow y = 2, -1, -3$.

Classify: Check sign of $f(y)$:



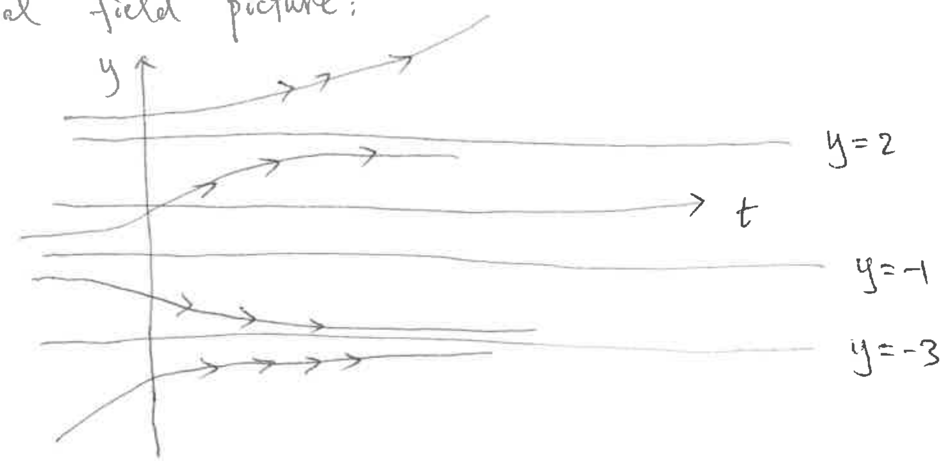
So



(stable) (unstable) (semi-stable).

$y=-3$ stable, $y=-1$ unstable, $y=2$ semi-stable.

Directional field picture:



Prob 7.

Find $Q(t)$.

Solution Let $Q(t)$ be amount of salt at time t.

Set up: $\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out}) = 5 \times 10 - \frac{Q}{200} \times 10$.

So $Q' = 50 - \frac{1}{20}Q$ or $Q' + \frac{1}{20}Q = 50$.

Then integrating factor $\mu(t) = e^{\int \frac{1}{20} dt} = e^{\frac{1}{20}t}$

and $Q(t) = \dfrac{\int e^{\frac{1}{20}t} \, 50 \, dt}{e^{\frac{1}{20}t}} = \dfrac{1000\, e^{\frac{1}{20}t} + C}{e^{\frac{1}{20}t}} = 1000 + C e^{-\frac{1}{20}t}$

Initially, $Q(0) = 3 \times 200 = 600$. plug in $t=0$:

$Q(0) = 1000 + C = 600 \Rightarrow C = -400$.

So $\boxed{Q(t) = 1000 - 400\, e^{-\frac{1}{20}t}}$

Prob. 8 $\left(2xy - \sin(x)\right) dx + (x^2 + 2y)\, dy = 0$.

<u>Solution</u>. $\begin{cases} M = 2xy - \sin(x) \\ N = x^2 + 2y \end{cases}$ check $\begin{cases} M_y = 2x \\ N_x = 2x \end{cases}$ ✓ exact!

Find $F(x,y) = \int M\, dx = \int (2xy - \sin(x))\, dx = x^2 y + \cos(x) + g(y)$.

Use $F_y = N$: $\frac{\partial}{\partial y}\left(x^2 y + \cos(x) + g(y) \right) = x^2 + 2y$.

$\Rightarrow \quad x^2 + g'(y) = x^2 + 2y$.

$\Rightarrow \quad g'(y) = 2y$ or $g(y) = y^2$.

So $F(x,y) = x^2 y + \cos(x) + y^2$ and solution is

$\boxed{x^2 y + \cos(x) + y^2 = C}$

explicit form: (quadratic formula) $y^2 + x^2 y + \cos(x) - C = 0$.

$\Rightarrow$. $\boxed{y = \dfrac{-x^2 \pm \sqrt{x^4 - 4\cos(x) + C}}{2}}$

Prob. 9 $\quad y'' - 7y' + 12y = 0.$

Solution. Characteristic eq. $r^2 - 7r + 12 = 0.$

Then $(r-3)(r-4) = 0$

$r_1 = 3$ and $r_2 = 4.$

So $y_1 = e^{3t}$ and $y_2 = e^{4t}.$

Then $\boxed{y = c_1 e^{3t} + c_2 e^{4t}.}$

Prob. 10 $\quad t^2 y'' - 2y = 0 \quad (t>0).$ Given $y_1 = t^2$ and $y_2 = t^{-1}.$

Solution. Verify: ① $y_1' = 2t$ and $y_1'' = 2.$

Plug back in: $t^2 y_1'' - 2y_1 = t^2 \cdot 2 - 2t^2 = 0$ ✓

② $y_2' = -t^{-2}$ and $y_2'' = 2t^{-3}$

Plug back in: $t^2 y_2'' - 2y_2 = t^2 \cdot 2t^{-3} - 2 \cdot t^{-1} = 0$ ✓

So y_1 and y_2 are solutions.

Check Wronskian: $W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det \begin{pmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{pmatrix}$

$= (-1) - 2 = -3 \neq 0.$

So $\{y_1, y_2\}$ is a set of fundamental solutions.

General solution $\boxed{y = c_1 t^2 + c_2 t^{-1}}$