Math 308 (WIR #6).

Prob 1. Find the form of the particular solution.

(a) $y'' - 3y' + 2y = -2t^2 e^{4t}$.

homogeneous eq. $y'' - 3y' + 2y = 0$. $\Rightarrow$ $r^2 - 3r + 2 = 0$.

So $(r-2)(r-1) = 0$ $\Rightarrow$ $r_1 = 2$, $r_2 = 1$.

$y_1 = e^{2t}$ and $y_2 = e^{t}$.

$g(t) = -2t^2 e^{4t}$ Same form: $\boxed{y_p = (At^2 + Bt + C)e^{4t}}$

(b). $y'' - 3y' + 2y = -2t^2 e^{2t}$.

from (a) $y_1 = e^{2t}$ and $y_2 = e^{t}$.

$g(t) = -2t^2 e^{2t}$ Same form. $y_p = (At^2 + Bt + C)e^{2t}$

But $C e^{2t}$ multiple of $y_1 = e^{2t}$. fix it: $\boxed{\tilde{y}_p = t(At^2 + Bt + C)e^{2t}}$.

(c). $y'' - 4y' + 4y = -2t^2 e^{4t}$

homogeneous eq. $y'' - 4y' + 4y = 0$. $\Rightarrow$ $r^2 - 4r + 4 = 0$.

So $(r-2)^2 = 0$. $\Rightarrow$ $r = 2$ (repeated).

$y_1 = e^{2t}$ and $y_2 = t e^{2t}$.

$g(t) = -2t^2 e^{4t}$ same form: $\boxed{y_p = (At^2 + Bt + C)e^{4t}}$

(d) $y'' - 4y' + 4y = -2t^2 e^{2t}$

from (c) $y_1 = e^{2t}$ and $y_2 = t e^{2t}$.

$g(t) = -2t^2 e^{2t}$ same form : $y_p = (At^2 + Bt + C) e^{2t}$.

But Ce^{2t} multiple of $y_1 = e^{2t}$, fix: $t(At^2 + Bt + C)e^{2t}$.

But $C \cdot t e^{2t}$ multiple of $y_2 = te^{2t}$ fix again :

$$\boxed{\tilde{y}_p = t^2 (At^2 + Bt + C) e^{2t}}$$

(e) $y'' - 4y' + 4y = 3e^{2t}\cos(2t)$.

from (c) $y_1 = e^{2t}$ and $y_2 = te^{2t}$.

$g(t) = 3e^{2t}\cos(2t)$. Same form: $\boxed{y_p = Ae^{2t}\cos(2t) + Be^{2t}\sin(2t)}$

(f) $y'' - 4y' + 13y = 3e^{2t}\cos(2t)$.

homogeneous eq. $y'' - 4y' + 13y = 0 \Rightarrow r^2 - 4r + 13 = 0$.

So $r = \dfrac{4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} = 2 \pm 3i$

$y_1 = e^{2t}\cos(3t)$ and $y_2 = e^{2t}\sin(3t)$

$g(t) = 3e^{2t}\cos(2t)$ same form: $\boxed{y_p = Ae^{2t}\cos(2t) + Be^{2t}\sin(2t)}$

(g). $y'' - 4y' + 13y = 3e^{2t}\cos(3t)$.

from (f) $\quad y_1 = e^{2t}\cos(3t)$ and $y_2 = e^{2t}\sin(3t)$.

$g(t) = 3e^{2t}\cos(3t)$ same form: $\quad y_p = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$.

But $Ae^{2t}\cos(3t)$ multiple of $y_1 = e^{2t}\cos(3t)$.

$Be^{2t}\sin(3t)$ multiple of $y_2 = e^{2t}\sin(3t)$.

fix it: $\boxed{\tilde{y}_p = t.\left[Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)\right]}$

(h). $y'' - 5y' = t^3$.

homogeneous eq. $y'' - 5y' = 0$. $\Rightarrow$ $r^2 - 5r = 0$.

So $r(r-5) = 0$ $\Rightarrow$ $r_1 = 0$ and $r_2 = 5$.

$y_1 = 1$ and $y_2 = e^{5t}$

$g(t) = t^3$ same form: $\quad y_p = At^3 + Bt^2 + Ct + D$.

But D is a multiple of $y_1 = 1$; fix it: $\boxed{\hat{y}_p = t(At^3 + Bt^2 + Ct + D)}$

Prob. 2. Undetermined Coefficients Solve ODEs:

(a) $y'' - 2y' - 3y = 3e^{2t}$

Step. <1>. $y'' - 2y' - 3y = 0$. $\Rightarrow$ $r^2 - 2r - 3 = 0$

So $(r-3)(r+1) = 0$ $\quad r_1 = 3 \quad r_2 = -1$.

$y_1 = e^{3t}$ and $y_2 = e^{-t}$

Step <2> : $g(t) = 3e^{2t}$, so $y_p = Ae^{2t}$ (not exception).

Compute $y_p' = 2Ae^{2t}$ and $y_p'' = 4Ae^{2t}$.

Plug back in $y_p'' - 2y_p' - 3y_p = 3e^{2t}$.

$$\Rightarrow \quad 4Ae^{2t} - 2\cdot 2Ae^{2t} - 3Ae^{2t} = 3e^{2t}.$$

$$\Rightarrow \quad -3Ae^{2t} = 3e^{2t} \quad \text{or} \quad A = -1.$$

So $y_p = -e^{2t}$

Step <3> : General solution $\boxed{y = c_1 e^{3t} + c_2 e^{-t} - e^{2t}}$

(b) $y'' - 2y' - 3y = -3te^{-t}$

Step <1>. from (a) $y_1 = e^{3t}$ and $y_2 = e^{-t}$

Step <2>. $g(t) = -3te^{-t}$, $y_p = (At+B)e^{-t}$. (exception)

fix : $y_p = t(At+B)e^{-t} = (At^2+Bt)e^{-t}$.

Compute
$$y_p' = (2At+B)e^{-t} + (At^2+Bt)\cdot(-1)e^{-t}$$
$$= [-At^2 + (2A-B)t + B]e^{-t}.$$

and
$$y_p'' = (-2At + 2A-B)e^{-t} + [-At^2 + (2A-B)t+B]\cdot(-1)e^{-t}.$$
$$= [At^2 + (-4A+B)t + (2A-2B)]e^{-t}$$

Plug back in : $y_p'' - 2y_p' - 3y_p = -3te^{-t}$

$$[At^2 + (-4A+B)t + (2A-2B)]e^{-t} - 2[-At^2 + (2A-B)t+B]e^{-t} - 3(At^2+Bt)e^{-t} = -3te^{-t}$$

$$\Rightarrow \quad (A+2A-3A)t^2e^{-t} + (-4A+B-4A+2B-3B)te^{-t} + (2A-2B-2B)e^{-t} = -3te^{-t}.$$

Then $\begin{cases} -8A = -3 \\ 2A - 4B = 0. \end{cases} \Rightarrow \begin{cases} A = \frac{3}{8} \\ B = \frac{3}{16}. \end{cases} \Rightarrow y_p = \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}.$

Step <3> : General solution $\boxed{y = c_1 e^{3t} + c_2 e^{-t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}}.$

(c). $y'' - 2y' - 3y = 3e^{2t} - 3te^{-t}.$

from (a) & (b). $\begin{cases} y'' - 2y' - 3y = 3e^{2t} \\ y'' - 2y' - 3y = -3te^{-t} \end{cases}$

$y_{p_1} = -e^{2t}$ and $y_{p_2} = \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}.$

So $y_p = y_{p_1} + y_{p_2} = -e^{2t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}.$

and general solution $\boxed{y = c_1 e^{3t} + c_2 e^{-t} - e^{2t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}.}$

Prob. 3. undetermined Coefficients. Solve $y'' - 2y' + y = te^t + 4$

with $y(0) = 1$ and $y'(0) = 1.$

Step <1> homogeneous eq. $y'' - 2y' + y = 0. \Rightarrow r^2 - 2r + 1 = 0.$

So $(r-1)^2 = 0 \Rightarrow r = 1$ (repeated)

$y_1 = e^t$ and $y_2 = te^t.$

Step <2>. $\begin{cases} y'' - 2y' + y = te^t \\ y'' - 2y' + y = 4. \end{cases}$ $\quad y_{p_1} = t^2(At + B)e^t = (At^3 + Bt^2)e^t$

$y_{p_2} = C$

Compute $y_{p_1}' = (3At^2+2Bt)e^t + (At^3+Bt^2)e^t = [At^3+(3A+B)t^2+2Bt]e^t$.

and $y_{p_1}'' = [3At^2+(6A+2B)t+2B]e^t + [At^3+(3A+B)t^2+2Bt]e^t$

$= [At^3+(6A+B)t^2+(6A+4B)t+2B]e^t$.

Plug back in: $y_{p_1}'' - 2y_{p_1}' + y_{p_1} = te^t$

$\Rightarrow e^t\{[At^3+(6A+B)t^2+(6A+4B)t+2B] - 2[At^3+(3A+B)t^2+2Bt] + (At^3+Bt^2)\} = te^t$

$\Rightarrow [(A-2A+A)t^3 + (6A+B-6A-2B+B)t^2 + (6A+4B-4B)t + 2B]\,e^t = te^t$.

$\Rightarrow \begin{cases} 6A = 1 \\ 2B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{6} \\ B = 0 \end{cases} \Rightarrow y_{p_1} = \frac{1}{6}t^3e^t$

Compute $y_{p_2}' = 0$ and $y_{p_2}'' = 0$.

Plug back in: $y_{p_2}'' - 2y_{p_2}' + y_{p_2} = 4$.

$\Rightarrow 0 - 2\cdot 0 + C = 4 \Rightarrow C = 4 \Rightarrow y_{p_2} = 4$.

Therefore $y_p = \frac{1}{6}t^3e^t + 4$.

Step (3). General solution $y = c_1e^t + c_2te^t + \frac{1}{6}t^3e^t + 4$.

Step (4). Use $y(0)=1$: $y(0) = c_1 + 4 = 1 \Rightarrow c_1 = -3$.

$y = -3e^t + c_2te^t + \frac{1}{6}t^3e^t + 4$ and $y' = -3e^t + c_2e^t + c_2te^t + \frac{1}{2}t^2e^t + \frac{1}{6}t^3e^t$

Use $y'(0)=1$: $y'(0) = -3 + c_2 = 1 \Rightarrow c_2 = 4$.

Hence $\boxed{y = -3e^t + 4te^t + \frac{1}{6}t^3e^t + 4.}$

Prob. 4. Variation of Parameters : find a particular solution

$$y'' - y' - 2y = 2e^{-t}.$$

Step (1). $y'' - y' - 2y = 0 \Rightarrow r^2 - r - 2 = 0$.

So $(r-2)(r+1) = 0 \Rightarrow r_1 = 2,\ r_2 = -1$.

$y_1 = e^{2t}$ and $y_2 = e^{-t}$.

Step (2). Let $y_p = u_1 y_1 + u_2 y_2$.

Compute $W(y_1, y_2) = \det\begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} = -e^{t} - 2e^{t} = -3e^{t} \neq 0$.

So $u_1 = \int \frac{-y_2 f(t)}{W}\,dt = \int \frac{-e^{-t} \cdot 2e^{-t}\,dt}{-3e^{t}} = \frac{2}{3}\int e^{-3t}\,dt = -\frac{2}{9}e^{-3t}$

and $u_2 = \int \frac{y_1 f(t)}{W}\,dt = \int \frac{e^{2t} \cdot 2e^{-t}}{-3e^{t}}\,dt = -\frac{2}{3}\int dt = -\frac{2}{3}t$.

(Choose constant to be 0).

Then $y_p = -\frac{2}{9}e^{-3t} \cdot e^{2t} + -\frac{2}{3}t\,e^{-t}$

$= \boxed{-\frac{2}{9}e^{-t} - \frac{2}{3}te^{-t}}$

Prob 5. $t y'' - (t+1)y' + y = t^2 e^{2t}$. $(t>0)$; $y_1 = t+1$ and $y_2 = e^{t}$.

Verify: $y_1' = 1$ and $y_1'' = 0$.

So $t y_1'' - (t+1)y_1' + y_1 = 0 - (t+1) + (t+1) = 0$ ✓

(homogeneous eq.)

$y_2' = e^{t}$ and $y_2'' = e^{t}$.

So $t y_2'' - (t+1)y_2' + y_2 = [t - (t+1) + 1]e^{t} = 0$ ✓ (homogeneous eq.).

Compute $W(y_1, y_2) = \det\begin{pmatrix} t+1 & e^t \\ 1 & e^t \end{pmatrix} = (t+1)e^t - e^t = te^t \neq 0$.

So $\{y_1, y_2\}$ fundamental solutions.

Let $y_p = u_1 y_1 + u_2 y_2$. and $y'' - \frac{t+1}{t}y' + \frac{1}{t}y = te^{2t}$ (standard)

Then $u_1 = \int \frac{-y_2 f(t)}{W}\,dt = \int \frac{-e^t \cdot te^{2t}}{te^t}\,dt = -\int e^{2t}\,dt$

$= -\frac{1}{2}e^{2t}$

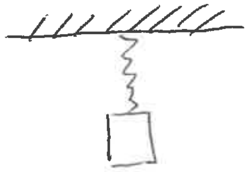
and $u_2 = \int \frac{y_1 f(t)}{W}\,dt = \int \frac{(t+1)\,te^{2t}}{te^t}\,dt = \int (t+1)e^t\,dt$.

$= e^t(t+1) - \int e^t \cdot 1\,dt = te^t$

So $y_p = -\frac{1}{2}e^{2t}(t+1) + te^t \cdot e^t = \frac{1}{2}te^{2t} - \frac{1}{2}e^{2t}$

General solution: $\boxed{y = c_1(t+1) + c_2 e^t + \frac{1}{2}te^{2t} - \frac{1}{2}e^{2t}}$

Prob. 6.



$m = 2\text{kg}$. $L = 0.98\,m$

$y(0) = 0.05\,m$. $y'(0) = 0.1\,m/s$

$my'' = -ky - cy' + F_{ext}$ no resistance: $c = 0$

no external: $F_{ext} = 0$.

Then $y'' + \omega_0^2 y = 0$ where $\omega_0^2 = \frac{k}{m}$.

Also stretch: $mg = kL$, So $k = \frac{mg}{L} = \frac{2 \times 9.8}{0.98} = 20\,N/m$

So $\omega_0^2 = \frac{20}{2} = 10 \ s^{-2}$ and $y'' + 10y = 0$.

Characteristic eq. $r^2 + 10 = 0 \Rightarrow r = \pm\sqrt{10}\,i$.

Then $y = c_1\cos(\sqrt{10}t) + c_2\sin(\sqrt{10}t)$.

Initial conditions: $y(0) = 0.05$ $\quad y'(0) = 0.1$

$y(0) = c_1 \Rightarrow c_1 = 0.05$.

Compute $y' = -\sqrt{10}c_1\sin(\sqrt{10}t) + \sqrt{10}c_2\cos(\sqrt{10}t)$

$y'(0) = \sqrt{10}c_2 \Rightarrow c_2 = \frac{0.1}{\sqrt{10}}$

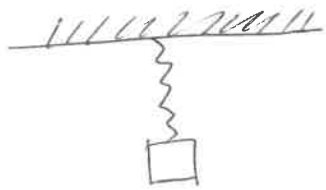
Therefore, $\boxed{y = 0.05\cos(\sqrt{10}t) + \frac{0.1}{\sqrt{10}}\sin(\sqrt{10}t)}$

Frequency: $\omega_0 = \sqrt{10} \ s^{-1}$.

Period: $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{10}} \ s$

Amplitude: $A = \sqrt{0.05^2 + \left(\frac{0.1}{\sqrt{10}}\right)^2} = \sqrt{0.0035} \ m$.

Prob. 7.



$m = 2kg \quad L = 0.98 \ m$

$y(0) = -0.1 \ m \quad y'(0) = -0.2 \ m/s$

$F_{ext} = 12\sin(2t) \ N$, $\quad F_d = 1.2N = \frac{c \cdot 10}{100}$

So $c = 12 \ \frac{N \cdot s}{m}$ and $k = \frac{mg}{L} = \frac{2 \times 9.8}{0.98} = 20 \ N/m$

Since $my'' = -ky - cy' + F_{ext}$,

$y'' + by' + \omega_0^2 y = f(t)$. where $b = \frac{12}{2} = 6$. $\omega_0^2 = \frac{k}{m} = 10$.

$f(t) = F_{ext}/m = 6\sin(2t)$.

So $y'' + 6y' + 10y = 6\sin(2t)$ with $y(0) = -0.1$ $y'(0) = -0.2$

Char. eq. $r^2 + 6r + 10 = 0.$ $\Rightarrow$ $r = \frac{-6 \pm \sqrt{6^2 - 40}}{2} = -3 \pm i$

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1 e^{-3t}\cos(t) + c_2 e^{-3t}\sin(t) + y_p.$$

Undetermined coefficients: $y_p = A\cos(2t) + B\sin(2t).$

Compute $y_p' = -2A\sin(2t) + 2B\cos(2t)$ and $y_p'' = -4A\cos(2t) - 4B\sin(2t).$

Plug back in: $y_p'' + 6y_p' + 10y_p = 6\sin(2t)$

$$\Rightarrow \left[-4A\cos(2t) - 4B\sin(2t)\right] + 6\left[-2A\sin(2t) + 2B\cos(2t)\right] + 10\left(A\cos(2t) + B\sin(2t)\right) = 6\sin(2t)$$

$$\Rightarrow \left[6A + 12B\right]\cos(2t) + \left[-12A + 6B\right]\sin(2t) = 6\sin(2t).$$

So $\begin{cases} 6A + 12B = 0 \\ -12A + 6B = 6 \end{cases}$ $\Rightarrow$ $\begin{cases} A = -\frac{2}{5} = -0.4 \\ B = \frac{1}{5} = 0.2 \end{cases}$

Then $y = c_1 e^{-3t}\cos(t) + c_2 e^{-3t}\sin(t) - 0.4\cos(2t) + 0.2\sin(2t).$

Use $y(0) = -0.1$ and $y'(0) = -0.2$:

$y(0) = c_1 - 0.4$ $\Rightarrow$ $c_1 = 0.3$

$$y' = -3c_1 e^{-3t}\cos(t) - c_1 e^{-3t}\sin(t) - 3c_2 e^{-3t}\sin(t) + c_2 e^{-3t}\cos(t) + 0.8\sin(2t) + 0.4\cos(2t)$$

$y'(0) = -3c_1 + c_2 + 0.4 = -0.9 + c_2 + 0.4 = -0.5 + c_2.$

$\Rightarrow$ $c_2 = 0.3$

Therefore $\boxed{y = 0.3e^{-3t}\cos(t) + 0.3e^{-3t}\sin(t) - 0.4\cos(2t) + 0.2\sin(2t)}$