Math 308 (WIR #7).

Prob 1. (a) $F(s) = \frac{3}{s^2+4}$.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \boxed{\frac{3}{2}\sin(2t).}$$

(b). $F(s) = \frac{1-2s}{s^2+4s+5}$

$$\mathcal{L}^{-1}\left\{\frac{1-2s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{-2(s+2)+5}{(s+2)^2+1}\right\}.$$

$$= \mathcal{L}^{-1}\left\{\frac{-2(s+2)}{(s+2)^2+1}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}.$$

$$= \boxed{-2e^{-2t}\cos(t) + 5e^{-2t}\sin(t).}$$

(c). $F(s) = \frac{2s-3}{s^2-4}$.

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s-2} + \frac{7/4}{s+2}\right\}$$

$$= \boxed{\frac{1}{4}e^{2t} + \frac{7}{4}e^{-2t}}$$

Here partial fractions $\frac{2s-3}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2}$.

$$\Rightarrow \quad 2s-3 = A(s+2) + B(s-2)$$
$$= (A+B)s + (2A-2B).$$

$$\Rightarrow \begin{cases} A+B = 2 \\ 2A-2B = -3 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = \frac{7}{4}. \end{cases}$$

(d) $$F(s) = \frac{8s^2-4s+12}{s(s^2+4)}.$$

Partial fractions : $$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

$$\Rightarrow \quad 8s^2-4s+12 = A(s^2+4) + s(Bs+C)$$
$$= (A+B)s^2 + Cs + 4A.$$

$$\Rightarrow \begin{cases} A+B=8 \\ C=-4 \\ 4A=12 \end{cases} \Rightarrow \begin{cases} A=3 \\ B=5 \\ C=-4 \end{cases} \Rightarrow \frac{8s^2-4s+12}{s(s^2+4)} = \frac{3}{s} + \frac{5s+(-4)}{s^2+4}$$

So $$\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5s-4}{s^2+4}\right\}$$

$$= 3 + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \boxed{3 + 5\cos(2t) - 2\sin(2t)}$$

Prob 2. (a). $$y'' + 3y' + 2y = e^{-t} \qquad y(0)=0, \quad y'(0)=1.$$

Apply Laplace transform to the ODE :

Compute $\mathcal{L}\{y\} = Y$, $\mathcal{L}\{y'\} = sY - y(0) = sY$. $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$
$= s^2Y - 1$

and $\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$

So $$(s^2Y - 1) + 3\cdot sY + 2Y = \frac{1}{s+1}$$

$$\Rightarrow \quad (s^2+3s+2)Y = 1 + \frac{1}{s+1} = \frac{s+2}{s+1}.$$

$$\Rightarrow \quad Y = \frac{s+2}{(s+1)(s^2+3s+2)} = \frac{s+2}{(s+1)(s+1)(s+2)} = \frac{1}{(s+1)^2}$$

Then $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$= \boxed{te^{-t}}$.

(b) $y'' + 9y = 3\sin(2t)$ $\quad y(0)=0$, $y'(0)=2$.

Apply Laplace transform to the ODE:

Compute $\mathcal{L}\{y\} = Y$. $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - 2$

and $\mathcal{L}\{3\sin(2t)\} = 3 \cdot \frac{2}{s^2+4} = \frac{6}{s^2+4}$.

So $s^2Y - 2 + 9Y = \frac{6}{s^2+4}$.

$\Rightarrow (s^2+9)Y = 2 + \frac{6}{s^2+4} = \frac{2s^2+14}{s^2+4}$

Then $Y = \frac{2s^2+14}{(s^2+4)(s^2+9)}$

Partial fractions: $\frac{2s^2+14}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$.

$\Rightarrow 2s^2+14 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$

$= (A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D)$.

$\Rightarrow \begin{cases} A+C=0 \\ 9A+4C=0 \end{cases}$ and $\begin{cases} B+D=2 \\ 9B+4D=14 \end{cases}$ $\Rightarrow \begin{cases} A=C=0. \\ B=\frac{6}{5} \\ D=4/5 \end{cases}$.

So $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{\frac{6}{5}}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{4}{5}}{s^2+9}\right\}$

$= \boxed{\frac{3}{5}\sin(2t) + \frac{4}{15}\sin(3t)}$

Prob 3. (a) $$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t-2 & 2 \le t < 4 \\ t^2-7t+14 & t \geq 4 \end{cases}$$

$$f(t) = 0 + [(t-2)-0]\,\mathcal{U}_2(t) + [(t^2-7t+14)-(t-2)]\,\mathcal{U}_4(t).$$

$$= (t-2)\,\mathcal{U}_2(t) + (t^2-8t+16)\,\mathcal{U}_4(t).$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)\,\mathcal{U}_2(t)\} + \mathcal{L}\{(t^2-8t+16)\,\mathcal{U}_4(t)\}.$$

$$= e^{-2s}\mathcal{L}\{t\} + e^{-4s}\mathcal{L}\{t^2\}$$

$$= \boxed{\frac{e^{-2s}}{s^2} + \frac{2e^{-4s}}{s^3}}$$

(b). $$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t-4 & 2 \le t < 4 \\ t^2-9t+5 & t \geq 4 \end{cases}$$

$$f(t) = 0 + [(t-4)-0]\,\mathcal{U}_2(t) + [(t^2-9t+5)-(t-4)]\,\mathcal{U}_4(t).$$

$$= (t-4)\,\mathcal{U}_2(t) + (t^2-10t+9)\,\mathcal{U}_4(t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-4)\,\mathcal{U}_2(t)\} + \mathcal{L}\{(t^2-10t+9)\,\mathcal{U}_4(t)\}.$$

$$= e^{-2s}\mathcal{L}\{t-2\} + \mathcal{L}\{[(t-5)^2-16]\,\mathcal{U}_4(t)\}.$$

$$= e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right) + e^{-4s}\mathcal{L}\{(t-1)^2-16\}.$$

$$= e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right) + e^{-4s}\mathcal{L}\{t^2-2t-15\}.$$

$$= \boxed{e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right) + e^{-4s}\left(\frac{2}{s^3} - \frac{2}{s^2} - \frac{15}{s}\right)}$$

Prob 4. (a). $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s}}{s}$

$$\mathcal{L}^{-1}\left\{ \frac{e^{-s}+e^{-2s}-e^{-3s}}{s} \right\} = \mathcal{L}^{-1}\left\{ \frac{e^{-s}}{s} \right\} + \mathcal{L}^{-1}\left\{ \frac{e^{-2s}}{s} \right\} - \mathcal{L}^{-1}\left\{ \frac{e^{-3s}}{s} \right\}$$

$$= u_1(t)\, \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\}\Big|_{(t-1)} + u_2(t)\, \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\}\Big|_{(t-2)} - u_3(t)\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\}\Big|_{(t}$$

$$= \boxed{u_1(t) + u_2(t) - u_3(t)}$$

(b) $F(s) = \dfrac{2}{s^2 - 4s + 13}$

$$\mathcal{L}^{-1}\left\{ \frac{2}{s^2-4s+13} \right\} = \mathcal{L}^{-1}\left\{ \frac{2}{(s-2)^2+9} \right\}$$

$$= 2e^{2t}\, \mathcal{L}^{-1}\left\{ \frac{1}{s^2+9} \right\}$$

$$= \boxed{\frac{2}{3} e^{2t} \sin(3t)}$$

(c) $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 4s + 13}$

$$\mathcal{L}^{-1}\left\{ \frac{2(s-1)e^{-2s}}{s^2-4s+13} \right\} = 2\, u_2(t)\, \mathcal{L}^{-1}\left\{ \frac{s-1}{(s-2)^2+9} \right\}\Big|_{(t-2)}$$

$$= 2u_2(t)\, e^{2(t-2)}\, \mathcal{L}^{-1}\left\{ \frac{s+1}{s^2+9} \right\}\Big|_{(t-2)}.$$

$$= \boxed{2u_2(t)\, e^{2(t-2)} \left(\cos[3(t-2)] + \frac{1}{3}\sin[3(t-2)] \right)}$$

Prob 5. Write $f(t) = 0 + 1 \cdot u_\pi(t) + (-1)u_{2\pi} = u_\pi(t) - u_{2\pi}(t)$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_\pi(t)\} - \mathcal{L}\{u_{2\pi}(t)\} = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

Compute $\mathcal{L}\{y\} = Y$, $\mathcal{L}\{y'\} = sY - y(0) = sY$, $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - 1$

So $(s^2Y - 1) + 2sY + 2Y = \mathcal{L}\{f(t)\} = \dfrac{e^{-\pi s} - e^{-2\pi s}}{s}$

$$\Rightarrow \quad (s^2+2s+2)Y = 1 + \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

$$\Rightarrow \quad Y = \frac{1}{s^2+2s+2} + \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}$$

Partial fractions: $\dfrac{1}{s(s^2+2s+2)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+2s+2}$

$$\Rightarrow \quad 1 = A(s^2+2s+2) + s(Bs+C) = (A+B)s^2 + (2A+C)s + 2A$$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A+C=0 \\ 2A=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=-1 \end{cases} \Rightarrow \frac{1}{s(s^2+2s+2)} = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}s+1}{s^2+2s+2}.$$

Then $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}\right\}$

$$= e^{-t}\sin(t) + u_{\pi}(t)\,g(t-\pi) - u_{2\pi}(t)\,g(t-2\pi).$$

where $g(t) = \mathcal{L}^{-1}\left\{\dfrac{\frac{1}{2}}{s} - \dfrac{\frac{1}{2}s+1}{s^2+2s+2}\right\} = \dfrac{1}{2} - \dfrac{1}{2}e^{-t}[\cos(t) + \sin(t)]$

Prob. 6. Compute $\mathcal{L}\{y\} = Y$, $\mathcal{L}\{y'\} = sY$, $\mathcal{L}\{y''\} = s^2Y$

and $\mathcal{L}\{\sin(t) - u_{2\pi}(t)\sin(t-2\pi)\} = \dfrac{1}{s^2+1} - e^{-2\pi s}\cdot\dfrac{1}{s^2+1} = \dfrac{1-e^{-2\pi s}}{s^2+1}$

So $s^2Y + 4Y = \dfrac{1-e^{-2\pi s}}{s^2+1} \Rightarrow Y = \dfrac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)}.$

Partial fractions: $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4} \Rightarrow 1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$

$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

So $\begin{cases} A+C=0 \\ 4A+C=0 \end{cases}$ and $\begin{cases} B+D=0 \\ 4B+D=1 \end{cases} \Rightarrow \begin{cases} A=C=0 \\ B=\frac{1}{3} \\ D=-\frac{1}{3}. \end{cases} \Rightarrow \dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{1}{3}\left(\dfrac{1}{s^2+1} - \dfrac{1}{s^2+4}\right).$

$$g(t) = \mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+4}\right\} = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)$$

Then $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{(1-e^{-2\pi s})G\} = g(t) - u_{2\pi}(t)\,g(t-2\pi)$

$$= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) - u_{2\pi}(t)\left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2(t-2\pi))\right]$$