Math 308 (WIR #8)

Prob 1. (a). $\int_{-\infty}^{\infty} \delta(t-3)\, dt$.

It's the total impulse of $\delta(t-3) = \delta_3(t)$

So $\int_{-\infty}^{\infty} \delta(t-3)\, dt = 1$.

Alternatively, $\int_{-\infty}^{\infty} \delta(t-3)\, dt = \int_{-\infty}^{\infty} \delta_3(t) \cdot 1\, dt = 1\Big|_{t=3} = 1$.

(b) $\int_{-\infty}^{\infty} \delta_3(t)\, t^2 \cos(t-2)\, dt$.

Use property : $\int_{-\infty}^{\infty} \delta_{t_0}(t)\, f(t)\, dt = f(t_0)$.

$\int_{-\infty}^{\infty} \delta_3(t)\, t^2 \cos(t-2)\, dt = \left[t^2 \cos(t-2)\right]\Big|_{t=3} = 3^2 \cos(3-2) = 9\cos(1)$.

(c). $\mathcal{L}\{(3t+1)\,\delta_3(t)\}$.

By definition, $\mathcal{L}\{(3t+1)\,\delta_3(t)\}$

$= \int_0^{\infty} e^{-st}(3t+1)\,\delta_3(t)\, dt$

$= \left[e^{-st}(3t+1)\right]\Big|_{t=3} = e^{-3s}(3\cdot 3+1) = \boxed{10e^{-3s}}$

Prob. 2. (a) Solve $y'' + y = \delta_{2\pi}(t)\cos(t)$ with $y(0)=0$ and $y'(0)=1$.

Apply Laplace transform: $\mathcal{L}\{y\} = Y$. $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$

$= s^2Y - 1$

and $\mathcal{L}\{\delta_{2\pi}(t)\cos(t)\} = \int_0^{\infty} e^{-st}\delta_{2\pi}(t)\cos(t)\, dt = e^{-st}\cos(t)\Big|_{t=2\pi} = e^{-2\pi s}$

Plug back in: $s^2Y - 1 + Y = e^{-2\pi s} \Rightarrow (s^2+1)Y = 1 + e^{-2\pi s}$

So $Y = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$ and $y = \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} + \mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s^2+1}\}$.

Therefore $y = \sin(t) + u_{2\pi}(t)\, \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}\Big|_{(t-2\pi)}$.

$= \boxed{\sin(t) + u_{2\pi}(t) \sin(t-2\pi)}$.

(b). Solve $y'' + 2y' + 2y = 3\delta_\pi(t)$. with $y(0)=1$, $y'(0)=0$.

Apply Laplace transform: $\mathcal{L}\{y\} = Y$, $\mathcal{L}\{y'\} = sY - y(0) = sY - 1$.

$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - s$ and $\mathcal{L}\{3\delta_\pi(t)\} = 3e^{-\pi s}$

Plug back in:

$$(s^2Y - s) + 2(sY - 1) + 2Y = 3e^{-\pi s}$$

$$\Rightarrow \quad (s^2+2s+2)Y = s+2 + 3e^{-\pi s}$$

So $Y = \frac{s+2}{s^2+2s+2} + \frac{3e^{-\pi s}}{s^2+2s+2}$.

Compute $\mathcal{L}^{-1}\{\frac{s+2}{s^2+2s+2}\} = \mathcal{L}^{-1}\{\frac{s+2}{(s+1)^2+1}\} = e^{-t}\mathcal{L}^{-1}\{\frac{s+1}{s^2+1}\} = e^{-t}[\cos(t) + \sin(t)]$

and $\mathcal{L}^{-1}\{\frac{3e^{-\pi s}}{s^2+2s+2}\} = 3u_\pi(t)\, \mathcal{L}^{-1}\{\frac{1}{s^2+2s+2}\}\Big|_{t-\pi} = 3u_\pi(t)\left[e^{-t}\mathcal{L}^{-1}\{\frac{1}{s^2+1}\}\right]\Big|_{t-\pi}$

$= 3u_\pi(t)\, e^{-(t-\pi)} \sin(t-\pi)$.

Therefore, $y = \mathcal{L}^{-1}\{Y\} = \boxed{e^{-t}[\cos(t) + \sin(t)] + 3u_\pi(t)\, e^{-(t-\pi)} \sin(t-\pi)}$

Prob. 3 Find $\mathcal{L}\{h(t)\}$, for $h(t) = \int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau$.

Recognize: $h(t) = (f * g)(t)$, where $f(t) = t^2$ and $g(t) = \cos(2t)$.

So $\mathcal{L}\{h(t)\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\} = \mathcal{L}\{t^2\}\cdot\mathcal{L}\{\cos(2t)\} = \frac{2}{s^3}\cdot\frac{s}{s^2+4} = \boxed{\frac{2}{s^2(s^2+4)}}$

Prob. 4. Find $\mathcal{L}^{-1}\left\{ \frac{s}{(s+1)(s^2+4)} \right\}$.

Recognize $\mathcal{L}^{-1}\left\{ \frac{s}{(s+1)(s^2+4)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \cdot \frac{s}{s^2+4} \right\}$.

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\} * \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} \right\}.$$

$$= \left(e^{-t}\right) * \left(\cos(2t)\right).$$

$$= \int_0^t e^{-(t-\tau)} \cos(2\tau)\, d\tau$$

$$= e^{-t} \int_0^t e^{\tau} \cos(2\tau)\, d\tau.$$

Note: $I = \int e^{\tau} \cos(2\tau)\, d\tau = \frac{e^{\tau}\sin(2\tau)}{2} - \int e^{\tau} \frac{\sin(2\tau)}{2}\, d\tau$

$$= \frac{1}{2} e^{\tau} \sin(2\tau) - \frac{1}{2}\left[e^{\tau} \frac{-\cos(2\tau)}{2} - \int e^{\tau} \frac{-\cos(2\tau)}{2}\, d\tau \right]$$

$$= \frac{1}{2} e^{\tau} \sin(2\tau) + \frac{1}{4} e^{\tau} \cos(2\tau) - \frac{1}{4} \int e^{\tau} \cos(2\tau)\, d\tau$$

$$= \frac{1}{2} e^{\tau} \sin(2\tau) + \frac{1}{4} e^{\tau} \cos(2\tau) - \frac{1}{4} I.$$

So $\frac{5}{4} I = \frac{1}{2} e^{\tau} \sin(2\tau) + \frac{1}{4} e^{\tau} \cos(2\tau) \Rightarrow I = \frac{2}{5} e^{\tau} \sin(2\tau) + \frac{1}{5} e^{\tau} \cos(2\tau)$.

Then $\mathcal{L}^{-1}\left\{ \frac{s}{(s+1)(s^2+4)} \right\} = e^{-t} \int_0^t e^{\tau} \cos(2\tau)\, d\tau$

$$= e^{-t} \cdot \left[\frac{2}{5} e^{\tau} \sin(2\tau) + \frac{1}{5} e^{\tau} \cos(2\tau) \right] \Bigg|_{\tau=0}^{\tau=t}$$

$$= e^{-t} \left[\frac{2}{5} e^{t} \sin(2t) + \frac{1}{5} e^{t} \cos(2t) - \frac{1}{5} \right]$$

$$= \boxed{\frac{2}{5} \sin(2t) + \frac{1}{5} \cos(2t) - \frac{1}{5} e^{-t}}$$

Prob. 5 Solve $y'' + 9y = h(t)$ with $y(0) = 0$, $y'(0) = 1$.

where $h(t) = \int_0^t \sin(2(t-\tau))\, \delta_4(\tau)\, d\tau$.

Apply Laplace transform: $\mathcal{L}\{y\} = Y$ and $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - 1$

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{\sin(2t) * \delta_4(t)\} = \mathcal{L}\{\sin(2t)\}\cdot\mathcal{L}\{\delta_4(t)\} = \frac{2}{s^2+4}\cdot e^{-4s}$$

Plug back in: $(s^2Y-1) + 9Y = \dfrac{2e^{-4s}}{s^2+4} \Rightarrow (s^2+9)Y = 1 + \dfrac{2e^{-4s}}{s^2+4}$.

So $Y = \dfrac{1}{s^2+9} + \dfrac{2e^{-4s}}{(s^2+4)(s^2+9)}$.

Partial fractions: $\dfrac{2}{(s^2+4)(s^2+9)} = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+9}$

$$\Rightarrow \quad 2 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$= (A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D)$$

So $\begin{cases} A+C=0 \\ B+D=0 \\ 9A+4C=0 \\ 9B+4D=2 \end{cases} \Rightarrow \begin{cases} A=C=0 \\ B=\frac{2}{5} \\ D=-\frac{2}{5} \end{cases} \Rightarrow \dfrac{2}{(s^2+4)(s^2+9)} = \dfrac{2}{5}\left(\dfrac{1}{s^2+4} - \dfrac{1}{s^2+9}\right)$.

Then $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\dfrac{2}{(s^2+4)(s^2+9)}\cdot e^{-4s}\right\}$.

$$= \frac{1}{3}\sin(3t) + u_4(t)\,\mathcal{L}^{-1}\left\{\frac{2}{5}\frac{1}{s^2+4} - \frac{2}{5}\frac{1}{s^2+9}\right\}\Bigg|_{(t-4)}$$

$$= \boxed{\frac{1}{3}\sin(3t) + u_4(t)\left[\frac{1}{5}\sin(2t-8) - \frac{2}{15}\sin(3t-12)\right]}$$

Prob 6. Solve $y'' + 3y' + 2y = \cos(t^2)$ $\quad y(0)=1,\ y'(0)=0$.

Apply Laplace transform: $\mathcal{L}\{y\} = Y$. $\mathcal{L}\{y'\} = sY - y(0) = sY - 1$.

$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y - s$. Let $g(t) = \cos(t^2)$, $\mathcal{L}\{g(t)\} = G(s)$.

Plug back in:

$$(s^2Y - s) + 3(sY-1) + 2Y = G(s)$$

$$\Rightarrow \quad (s^2+3s+2)Y = s+3 + G(s)$$

So $Y = \dfrac{s+3}{s^2+3s+2} + \dfrac{1}{s^2+3s+2}\cdot G(s)$.

Partial fractions: $\frac{s+3}{s^2+3s+2} = \frac{A}{s+2} + \frac{B}{s+1}$

$$\Rightarrow \quad s+3 = A(s+1) + B(s+2) = (A+B)s + (A+2B).$$

$$\begin{cases} A+B=1 \\ A+2B=3 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=2 \end{cases} \Rightarrow \frac{s+3}{s^2+3s+2} = \frac{-1}{s+2} + \frac{2}{s+1}$$

$$\frac{1}{s^2+3s+2} = \frac{C}{s+2} + \frac{D}{s+1}.$$

$$\Rightarrow \quad 1 = C(s+1) + D(s+2) = (C+D)s + (C+2D).$$

$$\begin{cases} C+D=0 \\ C+2D=1 \end{cases} \Rightarrow \begin{cases} C=-1 \\ D=1 \end{cases} \Rightarrow \frac{1}{s^2+3s+2} = \frac{-1}{s+2} + \frac{1}{s+1}$$

Then $y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+3s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\cdot G(s)\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{-1}{s+2} + \frac{2}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} * \mathcal{L}^{-1}\{G(s)\}$$

$$= -e^{-2t} + 2e^{-t} + \mathcal{L}^{-1}\left\{\frac{-1}{s+2} + \frac{1}{s+1}\right\} * (g(t)).$$

$$= -e^{-2t} + 2e^{-t} + \left(-e^{-2t} + e^{-t}\right) * \left(\cos(t^2)\right)$$

$$= \boxed{-e^{-2t} + 2e^{-t} + \int_0^t \left[-e^{-2(t-\tau)} + e^{-(t-\tau)}\right] \cos(\tau^2)\, d\tau}$$

Prob. 7. (a). $\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{k=0}^{\infty} a_k x^k.$

First $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

Then $x \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$

So $\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n$

$$= a_1 + \sum_{n=1}^{\infty} \left[(n+1) a_{n+1} + a_{n-1}\right] x^n$$

(b). $x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$.

First, $x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n$

Then $x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + \sum_{n=1}^{\infty} \left[(n+1) n a_{n+1} + a_n\right] x^n$$