



MATH 152 - WEEK-IN-REVIEW 1

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PROBLEM STATEMENTS

You should attempt the problems yourself first. The next section contains the solutions.

1. Apply u -substitution to evaluate the following integrals.

a. $\int x e^{7x^2} dx$

b. $\int \frac{x^2 + 2}{x^3 + 6x} dx$

c. $\int \frac{5 + 4x}{x^2 + 1} dx$

d. $\int x \sqrt{x + 5} dx$

e. $\int_0^1 5x^2 \sin(4x^3 - 7) dx$

f. $\int_1^2 x^3(1 - x^2)^5 dx$

2. Sketch the region enclosed by the curves and find its area.

$$y = 2x^2 + 5 \quad \text{and} \quad y = 5x^2 - 7$$

3. Sketch the region enclosed by the curves. Set up the integral with respect to both x and y that would give the area of the region.

$$y = x + 3 \quad \text{and} \quad y = \sqrt{2x + 6}$$

4. Find the area of the region in the first quadrant that is bounded by the curves:

$$xy = 12, \quad 3y = x \quad \text{and} \quad 3y = 4x.$$

5. Sketch the region bounded by the curve $y = e^{x/2}$, the tangent line to this curve at $x = 3$, the x -axis and the y -axis. Set up the integral(s) representing the area of this region.



SOLUTIONS

Click the boxed answer (also in red) to watch the video solution. Note any video errata. You can also see them all by viewing the [Week 1 playlist \(clickable link\)](#). You can turn on closed captions by clicking “CC” inside YouTube as well as adjust the video speed inside of “Settings” by clicking the cog in the bottom right of the player.

1. Apply u -substitution to evaluate the following integrals.

a. $\int x e^{7x^2} dx = \frac{1}{14} e^{7x^2} + C$

Solution. Set $u = 7x^2$ so that $du = 14x dx$ to get

$$\int x e^{7x^2} dx = \int e^{7x^2} (x dx) = \int e^u \left(\frac{1}{14} du \right) = \frac{1}{14} e^u + C = \frac{1}{14} e^{7x^2} + C.$$

b. $\int \frac{x^2 + 2}{x^3 + 6x} dx = \frac{1}{3} \ln |x^3 + 6x| + C$

Solution. Set $u = x^3 + 6x$ so that $du = (3x^2 + 6) dx = 3(x^2 + 2) dx$ to get

$$\int \frac{x^2 + 2}{x^3 + 6x} dx = \int \frac{(x^2 + 2) dx}{x^3 + 6x} = \int \frac{(1/3) du}{u} = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |x^3 + 6x| + C.$$

c. $\int \frac{5 + 4x}{x^2 + 1} dx = 5 \arctan(x) + 2 \ln |x^2 + 1| + C$

Solution. First split the integral into two pieces to get

$$\int \frac{5 + 4x}{x^2 + 1} dx = \int \frac{5}{x^2 + 1} dx + \int \frac{4x}{x^2 + 1} dx.$$

The first involves arctangent and u -substitution can be used in the second. Indeed, set $u = x^2 + 1$ so that $du = 2x du$ to get

$$\begin{aligned} \int \frac{5 + 4x}{x^2 + 1} dx &= \int \frac{5}{x^2 + 1} dx + \int \frac{4x}{x^2 + 1} dx = 5 \arctan(x) + \int \frac{2(2x dx)}{x^2 + 1} \\ &= 5 \arctan(x) + \int \frac{2 du}{u} = 5 \arctan(x) + 2 \ln |u| + C \\ &= 5 \arctan(x) + 2 \ln |x^2 + 1| + C. \end{aligned}$$



$$d. \int x\sqrt{x+5} dx = \boxed{\frac{10}{3}(x+5)^{3/2} - \frac{2}{5}(x+5)^{5/2} + C}$$

Solution. Set $u = x + 5$ so that $x = 5 - u$ and $du = dx$ to get

$$\begin{aligned} \int x\sqrt{x+5} dx &= \int (5-u)\sqrt{u} du = \int 5u^{1/2} - u^{3/2} du \\ &= \frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2} + C = \frac{10}{3}(x+5)^{3/2} - \frac{2}{5}(x+5)^{5/2} + C. \end{aligned}$$

$$e. \int_0^1 5x^2 \sin(4x^3 - 7) dx = \boxed{-\frac{5}{12} \cos(-3) + \frac{5}{12} \cos(-7)}$$

Solution. Set $u = 4x^3 - 7$ so that $du = 12x^2 dx$ and the limits are changed from $x = 0$ and $x = 1$ to $u = 4(0)^3 - 7 = -7$ and $u = 4(1)^3 - 7 = -3$, respectively. Computing:

$$\begin{aligned} \int_0^1 5x^2 \sin(4x^3 - 7) dx &= \int_0^1 \sin(4x^3 - 7) (5x^2 dx) = \int_{-7}^{-3} \sin(u) \left(\frac{5}{12} du\right) \\ &= \frac{-5}{12} \cos(u) \Big|_{-7}^{-3} = -\frac{5}{12} \cos(-3) + \frac{5}{12} \cos(-7) \end{aligned}$$

$$f. \int_1^2 x^3(1-x^2)^5 dx = \boxed{\frac{1}{2} \left[\frac{(-3)^7}{7} - \frac{(-3)^6}{6} \right]}$$

Solution. Set $u = 1 - x^2$ so that $x^2 = 1 - u$ and $du = -2x dx$. We'll drop the limits for now and deal with them later. Note that you could instead replace them using the definition of u if you prefer. We have:

$$\begin{aligned} \int x^3(1-x^2)^5 dx &= \int (1-x^2)^5(x^2)(x dx) = \int u^5(1-u) \left(-\frac{1}{2} du\right) \\ &= \frac{1}{2} \int u^6 - u^5 du = \frac{1}{2} \left(\frac{u^7}{7} - \frac{u^6}{6} \right) = \frac{1}{2} \left[\frac{(1-x^2)^7}{7} - \frac{(1-x^2)^6}{6} \right] \end{aligned}$$

Therefore, adding back in the limits gives

$$\int_1^2 x^3(1-x^2)^5 dx = \frac{1}{2} \left[\frac{(1-x^2)^7}{7} - \frac{(1-x^2)^6}{6} \right]_1^2 = \frac{1}{2} \left[\frac{(-3)^7}{7} - \frac{(-3)^6}{6} \right].$$



2. Sketch the region enclosed by the curves and find its area.

$$y = 2x^2 + 5 \quad \text{and} \quad y = 5x^2 - 7$$

The area is 32.

Solution. See the video for the sketch. We see that the top curve is $y = 2x^2 + 5$ and the bottom curve is $y = 5x^2 - 7$. Thus, the area between is given by

$$\begin{aligned} \int_{-2}^2 (2x^2 + 5) - (5x^2 - 7) dx &= \int_{-2}^2 -3x^2 + 12 dx = 2 \int_0^2 -3x^2 + 12 dx \\ &= 2 \left(-x^3 + 12x \right) \Big|_0^2 = 32, \end{aligned}$$

where the fact that $-3x^2 + 12$ is an even function was used.

3. Sketch the region enclosed by the curves. Set up the integral with respect to both x and y that would give the area of the region.

$$y = x + 3 \quad \text{and} \quad y = \sqrt{2x + 6}$$

The integrals giving the area are $\int_{-3}^{-1} \sqrt{2x + 6} - x - 3 dx$ and $\int_0^2 y - \frac{1}{2}y^2 dy$.

Solution. See the video for the sketch. We can find the points of intersection by setting $x + 3 = \sqrt{2x + 6} \Rightarrow x = -3, -1$ and noting $y = x + 3$ to get the points $(-3, -3 + 3) = (-3, 0)$ and $(-1, -1 + 3) = (-1, 2)$. This tells us the limits for the dx integral should go from -3 to -1 , while the limits for the dy integral should go from 0 to 2 . To get the dx integral, subtract the bottom curve from the top curve to get

$$\int_{-3}^{-1} \sqrt{2x + 6} - (x + 3) dx = \int_{-3}^{-1} \sqrt{2x + 6} - x - 3 dx.$$

To get the dy integral, first solve for y in each equation to get $y = x + 3 \Rightarrow x = y - 3$ and $y = \sqrt{2x + 6} \Rightarrow x = (1/2)y^2 - 3$. Now subtract the left curve from the right to get

$$\int_0^2 (y - 3) - \left(\frac{1}{2}y^2 - 3 \right) dy = \int_0^2 y - \frac{1}{2}y^2 dy.$$

Note that the dy integral is a bit easier to compute than the dx integral.



4. Find the area of the region in the first quadrant that is bounded by the curves:

$$xy = 12, \quad 3y = x \quad \text{and} \quad 3y = 4x.$$

The area is $12 \ln(2)$.

Video errata: You must set $x/3 = 4x/3$ to get the point $(0, 0)$. The video said setting these equal wouldn't give any new points, which is not true. There was also a minor typo when recording and I had to manually write 3 in $3y = 4x$.

Solution. See the video for the sketch. We can find the points of intersection by setting each curve equal to each other. First, solve for each one in terms of x to get

$$xy = 12 \Rightarrow y = 12/x, \quad 3y = x \Rightarrow y = x/3, \quad \text{and} \quad 3y = 4x \Rightarrow y = 4x/3.$$

Now setting the curves equal we have

$$12/x = x/3 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6,$$

$$12/x = 4x/3 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3 \quad \text{and}$$

$$x/3 = 4x/3 \Rightarrow x = 0.$$

We are only concerned with points in the first quadrant so plug in $x = 0, 3, 6$ into either of two corresponding curves we got them from to get the points

$$(0, 0/3) = (0, 0), \quad (3, 12/3) = (3, 4) \quad \text{and} \quad (6, 12/6) = (6, 2).$$

Looking at the graph, even though the bottom curve is always $y = x/3$, we see that we have different top curves depending on the x range. From $x = 0$ to 3 , the top curve is $y = 4x/3$, while the top curve from $x = 3$ to 6 is $y = 12/x$. This tells us to find the area, we should split this up into two integrals to get that the area is

$$\begin{aligned} \int_0^3 \frac{4x}{3} - \frac{x}{3} dx + \int_3^6 \frac{12}{x} - \frac{x}{3} dx &= \int_0^3 x dx + \int_3^6 \frac{12}{x} - \frac{x}{3} dx \\ &= \frac{x^2}{2} \Big|_0^3 + \left(12 \ln|x| - \frac{x^2}{6} \right) \Big|_3^6 \\ &= 12 [\ln(6) - \ln(3)] = 12 \ln(2), \end{aligned}$$

where the log rule $\ln(A) - \ln(B) = \ln(A/B)$ was used.



5. Sketch the region bounded by the curve $y = e^{x/2}$, the tangent line to this curve at $x = 3$, the x -axis and the y -axis. Set up the integral(s) representing the area of this region.

The area is represented by $\int_0^1 e^{x/2} dx + \int_1^3 e^{x/2} - \frac{e^{3/2}}{2}(x - 1) dx$.

Video errata: $e^{3x/2}$ was written in some spots where $e^{x/2}$ should have been written: when finding the point of intersection $(0, 1)$ and when writing the final integrals.

Solution. See the video for the sketch. To get the tangent line at $x = 3$, compute the derivative and evaluate there to get that the slope is

$$y'|_{x=3} = \frac{1}{2}e^{x/2}\Big|_{x=3} = \frac{1}{2}e^{3/2}.$$

Since it passes through $(3, e^{3/2})$, the equation of the tangent line is

$$y = \frac{1}{2}e^{3/2}(x - 3) + e^{3/2} = \frac{e^{3/2}}{2}(x - 1).$$

Looking at the graph, we see that $y = e^{x/2}$ is always the top curve, but this region should be split into two regions: one where the x -axis is the bottom curve and one where the tangent line is the bottom curve. To get where the tangent curve crosses the x -axis, set it equal to zero to get $x = 1$. Thus, the area is given by

$$\int_0^1 e^{x/2} - 0 dx + \int_1^3 e^{x/2} - \frac{e^{3/2}}{2}(x - 1) dx = \int_0^1 e^{x/2} dx + \int_1^3 e^{x/2} - \frac{e^{3/2}}{2}(x - 1) dx.$$