# Math 152 - Week-In-Review 3

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## Problem Statements

You should attempt the problems yourself first. The next section contains the solutions.

1. Evaluate each of the following using integration-by-parts.

   a. $\displaystyle\int x^2 e^{2x}\,dx$

   b. $\displaystyle\int (\ln(x))^2\,dx$

   c. $\displaystyle\int x^2 \tan^2(x)\,dx$

   d. $\displaystyle\int 6x^5 \cos(x^3)\,dx$

   e. $\displaystyle\int \sin(2x)e^{5x}\,dx$

2. Use trigonometric identities to evaluate the following integrals.

   a. $\displaystyle\int \tan^3(x)\,dx$

   b. $\displaystyle\int \frac{\sin^3(x)}{\sec^2(x)}\,dx$

   c. $\displaystyle\int \cos^2(x)\sin^2(x)\,dx$

   d. $\displaystyle\int \sec^4(x)\tan^3(x)\,dx$

   e. $\displaystyle\int \sin(3x)\cos(x)\,dx$

# Solutions

Click the boxed answer (also in red) to watch the video solution. Note any video errata. You can also see them all by viewing the Week 3 playlist (clickable link). You can turn on closed captions by clicking "CC" inside YouTube as well a adjust the video speed inside of "Settings" by clicking the cog in the bottom right of the player.

1. Evaluate each of the following using integration-by-parts.

a. $\int x^2 e^{2x}\,dx = \boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C}$

**Solution.** Use the tabular method by creating a column of derivatives of $u = x^2$ and another integrating $dv = e^{2x}$:

| D | +/− | I |
|---|---|---|
| $x^2$ | | $e^{2x}$ |
| | $+ \searrow$ | |
| $2x$ | | $\frac{1}{2}e^{2x}$ |
| | $- \searrow$ | |
| $2$ | | $\frac{1}{4}e^{2x}$ |
| | $+ \searrow$ | |
| $0$ | $\longrightarrow$ $-\int = C$ | $\frac{1}{8}e^{2x}$ |

$$\begin{aligned}\int x^2e^{2x}\,dx &= x^2\cdot\frac{1}{2}e^{2x} - 2x\cdot\frac{1}{4}e^{2x} + 2\cdot\frac{1}{8}e^{2x} + C\\ &= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C\end{aligned}$$

You could instead repeatedly use the $uv - \int v\,du$ formula, but this is generally faster.

b. $\int (\ln(x))^2\,dx = \boxed{x(\ln(x))^2 - 2x\ln(x) + 2x + C}$

**Solution.** First, rewrite the inside of the integral as $(\ln(x))^2 \cdot 1$ use the tabular method by creating a column of derivatives of $u = (\ln(x))^2$ and another integrating $dv = 1$:

| D | +/− | I |
|---|---|---|
| $(\ln(x))^2$ | $+$ ↘ | $1$ |
| $2\ln(x)\cdot\frac{1}{x}$ | $\longrightarrow$ $-\int$ | $x$ |

$$\begin{aligned}\int (\ln(x))^2\,dx &= (\ln(x))^2 \cdot x - \int 2\ln(x)\cdot\frac{1}{x}\cdot x\,dx \\ &= x(\ln(x))^2 - \int 2\ln(x)\,dx\end{aligned}$$

To compute the integral of $2\ln(x)$, similarly set $u = \ln(x)$ and $dv = 2$:

| D | +/− | I |
|---|---|---|
| $\ln(x)$ | $+$ ↘ | $2$ |
| $\frac{1}{x}$ | $\longrightarrow$ $-\int$ | $2x$ |

$$\begin{aligned}\int (\ln(x))^2\,dx &= x(\ln(x))^2 - \int 2\ln(x)\,dx \\ &= x(\ln(x))^2 - \left[\ln(x)\cdot 2x - \int \frac{1}{x}\cdot 2x\,dx\right] \\ &= x(\ln(x))^2 - 2x\ln(x) + \int 2\,dx \\ &= x(\ln(x))^2 - 2x\ln(x) + 2x + C\end{aligned}$$

c. $\int x^2 \tan^2(x)\,dx = \boxed{x\tan(x) - \frac{1}{2}x^2 + \ln|\cos(x)| + C}$

**Solution.** Use the tabular method by creating a column of derivatives of $u = x$ and another integrating $dv = \tan^2(x) = \sec^2(x) - 1$:

| D | +/− | I |
|---|---|---|
| $x$ | $+$ ↘ | $\tan^2(x) = \sec^2(x) - 1$ |
| $1$ | $\longrightarrow$ $-\int$ | $\tan(x) - x$ |

to get that

$$\begin{aligned}\int x^2 \tan^2(x)\,dx &= x\cdot(\tan(x) - x) - \int 1\cdot(\tan(x) - x)\,dx \\ &= x\tan(x) - x^2 - \int \tan(x)\,dx + \int x\,dx \\ &= x\tan(x) - x^2 - \int \frac{\sin(x)}{\cos(x)}\,dx + \frac{1}{2}x^2 \\ &= x\tan(x) - \frac{1}{2}x^2 - \int \frac{\sin(x)}{\cos(x)}\,dx.\end{aligned}$$

You can now use $u$-substitution with $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$ to get

$$\begin{aligned}\int x^2 \tan^2(x)\,dx &= x\tan(x) - \frac{1}{2}x^2 + \int \frac{1}{u}\,du \\ &= x\tan(x) - \frac{1}{2}x^2 + \ln|u| + C \\ &= x\tan(x) - \frac{1}{2}x^2 + \ln|\cos(x)| + C.\end{aligned}$$

d. $\int 6x^5 \cos(x^3)\,dx = \boxed{2x^3 \sin(x^3) + 2\cos(x^3) + C}$

**Solution.** First, apply substitution with $w = x^3$ to get

$$dw = 3x^2\,dx \quad \Rightarrow \quad 6x^5\,dx = (2x^3)(3x^2\,dx) = 2w\,dw$$

so that

$$\int 6x^5 \cos(x^3)\,dx = \int \cos(w) \cdot 2w\,dw,$$

where $w$ was used to avoid clashing of notation. This is not necessary if you do not make reference to $u$ in the next steps. Now use the tabular method by creating a column of derivatives of $u = 2w$ and another integrating $dv = \cos(w)$:

| D | +/− | I |
|---|---|---|
| $2w$ | | $\cos(w)$ |
| | $+$ ↘ | |
| $2$ | | $\sin(w)$ |
| | $-$ ↘ | |
| $0$ | $\longrightarrow$ | $-\cos(w)$ |
| | $+\int = C$ | |

$$\begin{aligned}
&\int 6x^5 \cos(x^3)\,dx \\
&= \int \cos(w) \cdot 2w\,dw \\
&= 2w \cdot \sin(w) - 2 \cdot (-\cos(w)) + C \\
&= 2x^3 \sin(x^3) + 2\cos(x^3) + C
\end{aligned}$$

e. $\int \sin(2x)e^{5x}\,dx = \boxed{\frac{5}{29}\sin(2x)e^{5x} - \frac{2}{29}\cos(2x)e^{5x} + C}$

**Solution.** This one will use circular integration-by-parts, with the goal to repeatedly apply integration-by-parts to end back up to a multiple of the original integral. It does not matter which one you choose to be $u$ or $dv$. Both will eventually give the same result. Let's use the tabular method by creating a column of derivatives of $u = \sin(2x)$ and another integrating $dv = e^{5x}$ (try swapping these for extra practice):

| D | +/− | I |
|---|---|---|
| $\sin(2x)$ | | $e^{5x}$ |
| | $+ \searrow$ | |
| $2\cos(2x)$ | | $\frac{1}{5}e^{5x}$ |
| | $- \searrow$ | |
| $-4\sin(2x)$ | $\longrightarrow$ $+\int = C$ | $\frac{1}{25}e^{5x}$ |

which gives

$$\begin{aligned}\int \sin(2x)e^{5x}\,dx &= \sin(2x)\cdot\frac{1}{5}e^{5x} - 2\cos(2x)\cdot\frac{1}{25}e^{5x} + \int -4\sin(2x)\cdot\frac{1}{25}e^{5x} \\ &= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x} - \frac{4}{25}\int \sin(2x)e^{5x}\,dx.\end{aligned}$$

Now set $A = \int \sin(2x)e^{5x}\,dx$ to get and solve for it to get

$$\begin{aligned} A &= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x} - \frac{4}{25}A \\ \Rightarrow\quad \frac{29}{25}A &= \frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x} + C \\ \Rightarrow\quad A &= \frac{25}{29}\left[\frac{1}{5}\sin(2x)e^{5x} - \frac{2}{25}\cos(2x)e^{5x}\right] + C \\ \Rightarrow\quad \int \sin(2x)e^{5x}\,dx &= \frac{5}{29}\sin(2x)e^{5x} - \frac{2}{29}\cos(2x)e^{5x} + C.\end{aligned}$$

You can wait until the final step to add the arbitrary constant $C$. Just don't forget it!

2. Use trigonometric identities to evaluate the following integrals.

a. $\int \tan^3(x)\,dx = \boxed{\frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C}$

**Solution.** First, split $\tan^3(x) = \tan(x)\cdot\tan^2(x)$ and use $\tan^2(x) = \sec^2(x) - 1$ to get

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \tan(x)\cdot\tan^2(x)\,dx = \int \tan(x)\left(\sec^2(x) - 1\right)\,dx\\ &= \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx.\\ &= \int \tan(x)\sec^2(x)\,dx + \int \frac{-\sin(x)}{\cos(x)}\,dx.\end{aligned}$$

This is now two substitutions. Use $u = \tan(x) \Rightarrow du = \sec^2(x)\,dx$ in the first integral and $v = \cos(x) \Rightarrow dv = -\sin(x)\,dx$ in the second. The use of two variables rather than just $u$ is to avoid mixing up the two different substitutions. Evaluating, we get

$$\begin{aligned}\int \tan^3(x)\,dx &= \int \tan(x)\sec^2(x)\,dx + \int \frac{-\sin(x)}{\cos(x)}\,dx\\ &= \int u\,du + \int \frac{dv}{v}\\ &= \frac{1}{2}u^2 + \ln|v| + C\\ &= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C.\end{aligned}$$

b. $\displaystyle\int \frac{\sin^3(x)}{\sec^2(x)}\,dx = \boxed{\frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C}$

**Video errata:** The rule of thumb is not always true. Rather than worrying about the size of the powers when dealing with powers of trig functions time powers of trig functions, it's better to consider which one can be reduced down in terms of the other after differentiating and then using trig identities. This often involves, but not always, the Pythagorean identities. You can then let $u$ be the trig function you can write the other in terms of. For this problem, by letting $u = \sin(x)$, you run into issues because you'll have to deal with a single (not raised to a power) $\cos(x)$ that's left over when writing $du$. You should try this yourself. However, sometimes either works just as well.

**Solution.** Begin by using rewriting the integral as

$$\int \frac{\sin^3(x)}{\sec^2(x)}\,dx = \int \sin^3(x)\cdot\cos^2(x)\,dx.$$

Now use $\sin^2(x) = 1 - \cos^2(x)$ and $u$-substitution with $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$ to get

$$\begin{aligned}\sin^3(x)\,dx &= \sin^2(x)\cdot\sin(x)\,dx = \left(1-\cos^2(x)\right)\cdot\sin(x)\,dx\\ &= (1-u^2)(-du) = (u^2-1)\,du\end{aligned}$$

and so

$$\begin{aligned}\int \frac{\sin^3(x)}{\sec^2(x)}\,dx &= \int \sin^3(x)\cdot\cos^2(x)\,dx = \int \cos^2(x)\left(\sin^3(x)\,dx\right)\\ &= \int u^2\left[(u^2-1)\,du\right] = \int u^4 - u^2\,du\\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C\\ &= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C.\end{aligned}$$

c. $\displaystyle\int \cos^2(x)\sin^2(x)\,dx = \boxed{\frac{x}{8} - \frac{1}{32}\sin(4x) + C}$

**Solution.** One approach is to first use $2\sin(x)\cos(x) = \sin(2x)$ to get

$$\begin{aligned}\int \cos^2(x)\sin^2(x)\,dx &= \int (\sin(x)\cos(x))^2\ dx = \int \left(\frac{1}{2}\sin(2x)\right)^2\ dx\\ &= \frac{1}{4}\int \sin^2(2x)\,dx.\end{aligned}$$

Now use $\sin^2(2x) = (1-\cos(4x))\,/2$ to get

$$\begin{aligned}\int \cos^2(x)\sin^2(x)\,dx &= \frac{1}{4}\int \sin^2(2x)\,dx\\ &= \frac{1}{4}\int \frac{1-\cos(4x)}{2}\,dx\\ &= \frac{1}{8}\int dx - \frac{1}{8}\int \cos(4x)\,dx\\ &= \frac{x}{8} - \frac{1}{32}\sin(4x) + C.\end{aligned}$$

*Alternative.* Another approach that you can use to practice is to apply the identities:

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \text{and} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

in the first step instead.

d. $\int \sec^4(x)\tan^3(x)\,dx = \boxed{\frac{1}{4}\tan^4(x) + \frac{1}{6}\tan^6(x) + C}$

**Solution.** Use $u$-substitution with $u = \tan(x) \Rightarrow du = \sec^2(x)\,dx$ along with the identity $\sec^2(x) = 1 + \tan^2(x)$ to get

$$\sec^4(x)\,dx = \sec^2(x)\left(\sec^2(x)\,dx\right) = \left(1+\tan^2(x)\right)\left(\sec^2(x)\,dx\right) = (1+u^2)\,du$$

and so

$$\begin{aligned}\int \sec^4(x)\tan^3(x)\,dx &= \int \tan^3(x)\left(1+\tan^2(x)\right)\left(\sec^2(x)\,dx\right)\\ &= \int u^3(1+u^2)\,du = \int u^3 + u^5\,du\\ &= \frac{1}{4}u^4 + \frac{1}{6}u^6 + C\\ &= \frac{1}{4}\tan^4(x) + \frac{1}{6}\tan^6(x) + C.\end{aligned}$$

*Alternative.* Another approach that you can use to practice is to set $u = \sec(x)$ instead. You will not get tangents in your answer, so why is it still correct? How is this answer related to the one above?

e. $\int \sin(3x)\cos(x)\,dx = \boxed{-\frac{1}{8}\cos(4x) - \frac{1}{4}\cos(2x) + C}$

**Solution.** Use the identity

$$\sin(3x)\cos(x) = \frac{1}{2}\left[\sin(3x+x) + \sin(3x-x)\right] = \frac{1}{2}\left[\sin(4x) + \sin(2x)\right]$$

to get

$$\begin{aligned}\int \sin(3x)\cos(x)\,dx &= \frac{1}{2}\int \sin(4x) + \sin(2x)\,dx \\ &= \frac{1}{2}\left[\frac{-\cos(4x)}{4} - \frac{\cos(2x)}{2}\right] + C \\ &= -\frac{1}{8}\cos(4x) - \frac{1}{4}\cos(2x) + C.\end{aligned}$$

*Alternative.* Try using circular integration-by-parts instead. The computation is more involved, but does not require the obscure trig identity.

 