Chapter 14 Partial Derivatives

Section 14.1 Functions of Several Variables

Concepts

• The definition of a function of two variables

• The graph of a function of two variables with domain D and range R

• The level curves of a function of two variables

Problem set for video

Consider $f(x,y) = \ln(y - 4x)$

1. Evaluate $f(-1,3)$.
2. Find the domain of $f(x,y)$ and sketch the domain of $f(x,y)$ in the xy-plane.

Find the domain of $f(x,y) = \dfrac{\sqrt{9 - x^2 - y^2}}{x + y}$ and sketch the domain of $f(x,y)$ in the xy-plane.

Sketch the level curves for $f(x,y) = 1 - 5x + y$ for $k = 1, 0, -1$.

Sketch the level curves for f(x,y)=$\sqrt{4 - x^2 - y^2}$ for k= 0, 1, 2.

Sketch the level curve for $f(x,y) = \sqrt{y^2 - x^2}$ for $k = 4$.

Section 14.2 Limits and Continuity

Concepts

• Calculating the limit of a surface

• The definition of the limit of a two-variable function

• Limits at infinity and infinite limits of two-variable functions

Problem set for video

Find the limit. If the limit does not exist, support your answer.

(a) $$\lim_{(x,y)\to(1,2)} \ln\left(1 + x^2y^2\right)$$

(b) $$\lim_{(?x,y)\to(0,0)} \frac{x^3 + xy^2}{x^2 + y^2}$$

Find the limit. If the limit does not exist, support your answer.

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2 + y^2 + 1} - 1}{x^2 + y^2}$$

Find the limit. If the limit does not exist, support your answer.

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$$

Find the limit. If the limit does not exist, support your answer.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Section 14.3 Partial Derivatives

Concepts

- The definition the partial derivative of $f(x, y)$ with respect to x and y
- The geometric interpretation of the partial derivative
- Higher order partial derivatives and Clairaut's Theorem

Problem set for video

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret them as slopes

If $f(x, y) = y\sin(-x) + 2y$, find $\left.\dfrac{\partial f}{\partial x}\right|_{(\pi,0)}$ and $\left.\dfrac{\partial f}{\partial y}\right|_{(\pi,0)}$

Find $f_y(x, y)$ if $f(x, y) = e^{\tan(xy^3)}$

Find all higher order partial derivatives of $f(x, y) = xe^y + y^2e^{6x}$

Section 14.4 Tangent Planes and Linear Approximations

Concepts

- The equation of the tangent plane
- Differentials
- Applications of differentials

Problem set for video

Find the equation of the the tangent plane to the surface $z = x^2 + 3y^2$ at the point $(1, -1, 4)$. What is the equation of the normal line to the surface at the point $(1, -1, 4)$?

Find the differential, dz, if $z = f(x, y) = x^2 + y^2$. If x changes from 2 to 2.5 and y changes from 3 to 2.96, compare the values of Δz and dz.

Use differentials to approximate $\sqrt{9(1.95)^2 + (8.1)^2}$

The length and width of a rectagle are measured to be 25 cm and 35 cm, respectively, with an error in measurement of at most 0.1 in the length and 0.2 in the width. Use differentials to estimate the maximum error in the calculated area of the rectangle.

Section 14.5 The Chain Rule

Concepts

- The chain rule for functions of more than one variable

• Related rates

Problem set for video

If $z = \ln(x^2 + y^2)$, $x = 1 + e^{6t}$, $y = \sec^2(5t)$, find $\dfrac{dz}{dt}$

If $z = x^2 - 3x^2y^3$, $x = se^t$, $y = se^{-t}$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

If $u = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2s\sin t$, find $\dfrac{\partial u}{\partial t}$ when $r = 2$, $s = 3$ and $t = 0$

The radius of a circular cone is increasing at a rate of 1.8 inches per second, while the height is decreasing at a rate of 2.5 inches per second. At what rate is the volume of the cone changing when the radius is 120 inches and the height is 140 inches?

Section 14.6 Directional Derivatives and The Gradient Vector

Concepts

• The Directional Derivative and can this be typed up: The **directional derivative** of $z = f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle$.

• The gradient vector

Problem set for video

Find $D_{\mathbf{u}}f(x, y)$ at the point $(1, 2)$ in the direction of $\langle 1, -3\rangle$ to the surface $f(x, y) = x^3 + 2x^2y^2$

Suppose $f(x, y) = x^3 - 2xy + y^2$. Find $D_{\mathbf{u}}f(x, y)$ at the point $(1, 2)$ where $\mathbf{u}$ is the unit vector corresponding to $\pi 3$

If $f(x, y, z) = z^3 - x^2y$, find $D_{\mathbf{u}}f(1, 6, 2)$ if $\mathbf{u} = \left\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right\rangle$

Let $f(x, y) = xe^y$

a.) Find the rate of change of f at the point $(2, 0)$ in the direction of the point $P(2, 0)$ to the point $Q\,(12, 2)$.

b.) At the point $(2, 0)$, in what direction does f have the maximum rate of change? What is the maximum rate of change?

Find the maximum rate of change of $f(x, y) = \tan(3x + 2y)$ at the point $\left(\dfrac{\pi}{6}, -\dfrac{\pi}{8}\right)$ and the direction in which it occurs.

Section 14.7 Maximum and Minimum Values

Concepts

• Local and absolute extrema of a function $z = f(x, y)$

A function $f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \le f(a, b)$ for all (x, y) in a disk with center (a, b). In this case we call $f(a, b)$ a **local maximum value** of $f(x, y)$. Similarly, $f(x, y)$ has a **local minimum** at (a, b) if $f(x, y) \ge f(a, b)$ for all (x, y) in a disk with center (a, b). In this case we call $f(a, b)$ a **local**

minimum value of $f(x, y)$.

A function $f(x, y)$ has an **absolute maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in the domain of f. In this case we call $f(a, b)$ the **absolute maximum value** of $f(x, y)$. Similarly, $f(x, y)$ has an **absolute minimum** at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f. In this case we call $f(a, b)$ the **absolute minimum value** of $f(x, y)$.

• **The Second Derivative Test for Local Extrema:** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$.

a.) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

b.) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

c.) If $D < 0$, then $f(a, b)$ is called a saddle point.

d.) If $D = 0$, the second derivative test gives no information.

• **Extreme Value Theorem for Functions of Two Variables**: If f is continuous on a closed, bounded set D in R^2, then f attains an absolute maximum and an absolute minimum on D

Problem set for video

Find all local extrema or saddle points for $f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$

Find all local extrema or saddle points for $f(x, y) = x^3 + 6xy - 2y^2$

A box with no lid is to hold 10 cubic meters. Find the dimensions of the box with a minimum surface area.

Find the absolute extrema of $f(x, y) = x^2 + y^2 - 2x$ on the closed triangular region with vertices $(2, 0)$, $(0, 2)$, $(0, -2)$.

Section 14.8 Lagrange Multipliers

Concepts

Let f and g have continuous first order partial derivatives such that f has an extremum at a point (x_0, y_0) on the constraint curve $g(x, y) = k$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a number λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$.

To find the absolute maximum or absolute minimum values of $z = f(x, y)$ subject to the constraint $g(x, y) = k$:

1. Find all values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

   and

$$g(x, y) = k$$

2. Evaluate f at all points (x, y) that arise from the previous step. The largest of these values is the

absolute maximum of f and the smallest of these values is the absolute minimum of f. Note: A similar procedure is followed for functions of three variables.

Problem set for video

Find the extreme values of $f(x, y) = 3x + y$ subject to the constraint $x^2 + y^2 = 10$.

Find the extreme values of $f(, y) = x^2 + 2y^2$ subject to the constraint $x^2 + 16y^2 = 16$.

Find the minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + 3y - 2z = 12$.

Find the volume of the largest rectangular box with faces parallel to the coordinate planes that can be inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$.