Directions

There are 7 questions below; do each of them. To receive credit, you must give an argument for why your answer is correct. This can be in the form of "showing your work". But you must demonstrate that you understand the material well enough to teach it to other students. All questions are graded out of 3 points. You receive 3 points if it is clear from your work that you have mastered the relevant material (this is usually more than just getting the "correct answer".) You receive 0 points if it is clear you don't understand the relevant material at all. Partial credit is awarded for solutions in between these two extrema.

Question	Score
1	
2	
3	
4	
5	
6	
7	
Total	

Question 7

Let $\boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

(a) Can the vector $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ be written as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$? If so, what are the coefficients? If not, give a short argument justifying this.

(b) Can the vector $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ be written as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$? If so, what are the coefficients? If not, give a short argument justifying this.

Question 2

(a) Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be three vectors in $\mathbb{R}^n$ such that $2\boldsymbol{u} + 3\boldsymbol{v} = \boldsymbol{w}$. Assume further that $\boldsymbol{u} + 2\boldsymbol{v} = \mathbf{0}$ (the zero vector). Find two numbers $a \neq 2$ and $b \neq 3$ such that $a\boldsymbol{u} + b\boldsymbol{v} = \boldsymbol{w}$.

(b) Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be three vectors in $\mathbb{R}^n$ such that $2\boldsymbol{u} + 3\boldsymbol{v} = \boldsymbol{w}$. Assume further that there are *no* non–zero scalars c_1, c_2 such that $c_1\boldsymbol{u} + c_2\boldsymbol{v} = \mathbf{0}$. Then give a short argument explaining why there are no non–zero scalars $a \neq 2$ and $b \neq 3$ such that $a\boldsymbol{u} + b\boldsymbol{v} = \boldsymbol{w}$.

Question 3

Let L be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad \text{and} \quad L\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

(a) Compute $L\begin{pmatrix}1\\0\end{pmatrix}$ and $L\begin{pmatrix}0\\1\end{pmatrix}$.

(b) Find the matrix representation of this linear transformation.

(c) Can you solve the equation $L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\0\\3\end{pmatrix}$? If so, what is the solution? If not, explain why not?

(d) Can you solve the equation $L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\1\\3\end{pmatrix}$? If so, what is the solution? If not, explain why not?

Question 4

Let $\mathcal{S} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \right\}$. Answer the following questions and be sure to justify your answers:

(a) Is $\mathcal{S}$ a linearly independent set?

(b) Is $\mathcal{S}$ a spanning set?

(c) Is $\mathcal{S}$ a basis?

Question 5

Consider the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ given by:

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 & 5 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Answer the following questions and be sure to justify your answers:

(a) Is L injective?

(b) Is L surjective?

(c) Is L a bijective?

(d) Find a basis for the range of L.

(e) Find all solutions to the equation: $L\boldsymbol{v} = \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}$.

Question 6

Let $\mathcal{S} = \mathcal{L}\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\\2\end{pmatrix}\}$. Find the projection of $\begin{pmatrix}1\\1\\1\\2\end{pmatrix}$ on to $\mathcal{S}$.

Question 7

Let $M = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & 0 \\ 2 & -2 & 0 \end{pmatrix}$.

(a) Find the eigenvalues of M.

(b) What is the *algebraic* multiplicity of each eigenvalue above?

(c) Find a basis for each eigenspace.