2024 Fall Math 140 Week-In-Review

Week 10: Sections 5.5 and 5.6

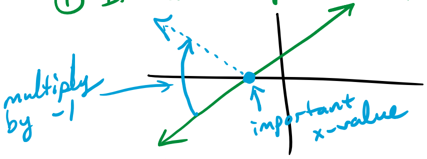
Some Key Words and Terms: Piecewise Functions, Absolute Value Functions, Domain of a Piecewise Functions, Exponential Expressions, Exponential Functions, Growth vs. Decay, Rewriting Bases.

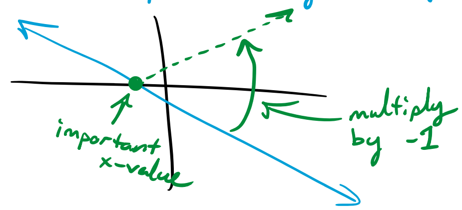
Piecewise Functions: Literally a function that is defined by separate pieces & those pieces correspond to specific ranges of x-values.

Absolute Value Function: Every absolute value function is really just a piecewise function. Linear absolute functions, will work in one of two ways:

① If absolute value part has positive slope

② If absolute value part has negative slope





Domain of a Piecewise Function:

★ this is an involved process

① work within each piece by finding the intersection of the domain w/ the given range of x-values

② find the union of the results from each piece

Exponential Expressions: any expression w/ a numerical base, b, with a variable in the exponent AND $b > 0$

★ we will never have negative bases in Math 140 or 142 ★

$2^{(x-1)}$ $\left(\frac{5}{6}\right)^{(7-x)}$ e^{2x-1} ....

$e \approx 2.7$-ish

Exponential Functions: functions defined by exponential expressions

★ again, base cannot be negative or zero ★

★ w/ functions, we also disregard a base of 1 ★

b/c $1^{(\text{number})} = 1$

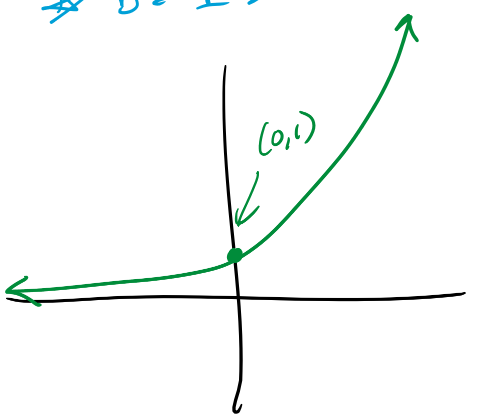
$f(x) = a \cdot b^x$

coefficient (a) — base of the function (b)

Growth vs. Decay:

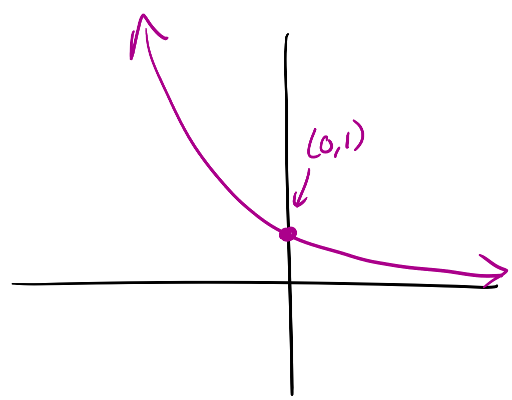
Growth: $f(x) = a \cdot b^x$

★ $b > 1$ ★ i.e. $f(x) = 2^x$



Decay: $f(x) = a \cdot b^x$

★ $0 < b < 1$ ★ i.e. $f(x) = \left(\frac{1}{2}\right)^x$



D: $(-\infty, \infty)$ y-int: $(0,1)$

R: $(0, \infty)$ x-int: none

as $x \to -\infty$, $y \to 0$

as $x \to \infty$, $y \to \infty$

★ neither one will ever equal zero ★

as $x \to -\infty$, $y \to \infty$

as $x \to \infty$, $y \to 0$

Rewriting Bases:

- Rewriting the numerical bases in simplest numerical form
- often the case that we want to combine exponential terms

★ to combine exponential terms, they must have the same base

$8^x \to (2^3)^x \to 2^{3x}$

$25^x \to (5^2)^x = 5^{2x}$

$27^x \to (3^3)^x \to 3^{3x}$

$\left(\frac{1}{4}\right)^x = \left(\frac{1}{2^2}\right)^x = \frac{1}{2^{2x}}$ ...

Examples:

1. For the given piecewise functions, determine the given value of the function, if it exists.

   (a) $g(x) = \begin{cases} \dfrac{1}{x+8} & \text{if } x \le -7 \quad ① \\ xe^x & \text{if } -7 < x < -1 \quad ② \\ 3x^2 + 2x - 1 & \text{if } x \ge 1 \quad ③ \end{cases}$

   i. $f(-9) =$ $x=-9$ $\quad -9 \le -7 \checkmark$ so $f(-9) = \frac{1}{-9+8} = \boxed{-1}$

   ii. $f(-7) =$ $x=-7$ $\quad -7 \le -7 \checkmark$ so $f(-7) = \frac{1}{-7+8} = \boxed{1}$

   iii. $f(0) =$ $x=0$ $\quad \left.\begin{array}{l} 0 \le -7 \ \times \\ -7 < 0 < -1 \ \times \\ 0 \ge 1 \ \times \end{array}\right\}$ if $x=0$ is not included in any interval, then $\boxed{f(0) \text{ DNE}}$

   (b) $k(x) = \begin{cases} \sqrt{7-4x} & \text{if } x \le -3 \leftarrow x=-5 \\ \dfrac{x(x-1)}{(x-1)(x+1)} & \text{if } -3 < x < 3 \leftarrow x=0 \\ 12\left(\dfrac{1}{2}\right)^x & \text{if } x \ge 3 \leftarrow x=3 \end{cases}$

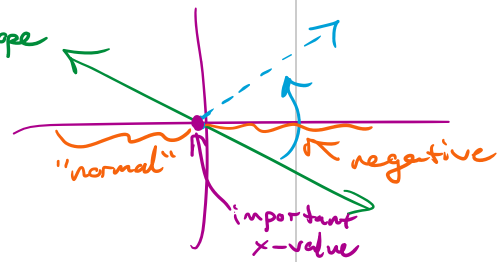
   i. $f(-5) = \sqrt{7-4(-5)} = \sqrt{7+20} = \overbrace{\sqrt{27} = 3\sqrt{3}}^{\text{depends on professor}}$

   ii. $f(0) = \frac{(0)(0-1)}{(0-1)(0+1)} = \frac{0}{-1} = \boxed{0}$

   iii. $f(3) = 12 \cdot \left(\frac{1}{2}\right)^3 = 12 \cdot \frac{1}{8} = \frac{12}{8} = \boxed{\frac{3}{2}}$

2. For the given absolute value functions, rewrite them as piecewise functions.

(a) $f(x) = |7 - 3x|$ 7-3x is a line w/ negative slope



① $7 - 3x = 0$

$-3x = -7$

$x = \frac{-7}{-3} = \frac{7}{3}$

② #-line

$+$ | $-$

test $x=0$ | $\frac{7}{3}$ | test $x=3$

$7-3(0)=7>0$ | $7-3(3)=-2<0$

$-\infty$ to $7/3$ positive | $\frac{7}{3}$ to ∞ negative

③ $$f(x) = \begin{cases} 7-3x, & x < 7/3 \\ -(7-3x), & x \geq 7/3 \end{cases}$$

doesn't matter which has "=" part as long as only one part has it

(b) $h(x) = 4|2x + 7| + 5$

① $2x + 7 = 0$

$2x = -7$

$x = -7/2$

② #-line

$-$ | $+$

$x=-4$ | $-7/2$ | $x=0$

$2(-4)+7=-1<0$ | $2(0)+7=7>0$

③ $$h(x) = \begin{cases} 4 \cdot -(2x+7)+5, & x < -\frac{7}{2} \\ 4 \cdot (2x+7)+5, & x \geq -\frac{7}{2} \end{cases}$$

(depending on prof) this could be enough for free response, but MC would be simplied

3. For the following piecewise function, determine the domain. Give your answer in interval notation.

3 domain restrictions:
1. denominator $(\neq 0)$
2. even root (inside ≥ 0)
3. log (inside > 0)

$$k(x) = \begin{cases} \sqrt{7-4x} & \text{if } x \leq -3 \\ \dfrac{x(x-1)}{(x-1)(x+1)} & \text{if } -3 < x < 3 \\ 12\left(\dfrac{1}{2}\right)^x & \text{if } x \geq 3 \end{cases}$$

$y_1 = \sqrt{7-4x}$

$y_2 = \dfrac{x(x-1)}{(x-1)(x+1)}$

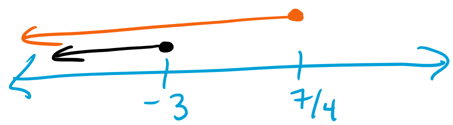
$y_3 = 12 \cdot \left(\frac{1}{2}\right)^x$

① $y_1 = \sqrt{7-4x}$

$7 - 4x \geq 0$

$x \leq 7/4$

D: $(-\infty, 7/4] \cap \underset{x \leq -3}{(-\infty, -3]}$



$(-\infty, -3]$ ☆

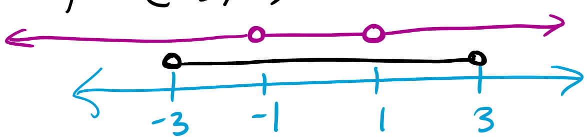
② $y_2 = \dfrac{x(x-1)}{(x-1)(x+1)}$ always find domain without cancel

$(x-1)(x+1) \neq 0$

$x - 1 \neq 0, \; x + 1 \neq 0$

$x \neq 1, \; x \neq -1$

D: $\left((-\infty,-1)\cup(-1,1)\cup(1,\infty)\right) \cap \underset{-3 < x < 3}{(-3, 3)}$

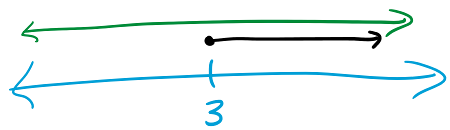


$(-3,-1)\cup(-1,1)\cup(1,3)$ ☆

③ $y_3 = 12 \cdot \left(\frac{1}{2}\right)^x$

no restrictions

D: $\underbrace{(-\infty, \infty)}_{\text{if } \mathbb{R}, \text{ then}} \cap \underbrace{\overset{x \geq 3}{[3, \infty)}}_{\text{is the result}}$



$[3, \infty)$ ☆

Domain is the union of all the blue results:

$(-\infty, -3] \cup (-3, -1) \cup (-1, 1) \cup (1, 3) \cup [3, \infty)$

☆ if we have a union between two of the same number with at least one include, then we don't need to break at that number ☆

$$\boxed{D: \; (-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$$

4. Graph the given piecewise function.

$$k(x) = \begin{cases} -9 - 3x & \text{if } -6 \le x < -3 \\ -2x^2 + 4x & \text{if } -3 \le x < 2 \\ -1 & \text{if } 4 < x < 8 \end{cases}$$

open!

$y_1 = -9 - 3x$ (line)

$y_2 = -2x^2 + 4x$ (parabola)

$y_3 = -1$ (horizontal line)

y_1 (since a line), just plug-in the start & stop x-values

$y_1(-6) = -9 - 3(-6) = -9 + 18 = 9$ closed $(-6, 9)$

$y_2(-3) = -9 - 3(-3) = -9 + 9 = 0$ $(-3, 0)$ open

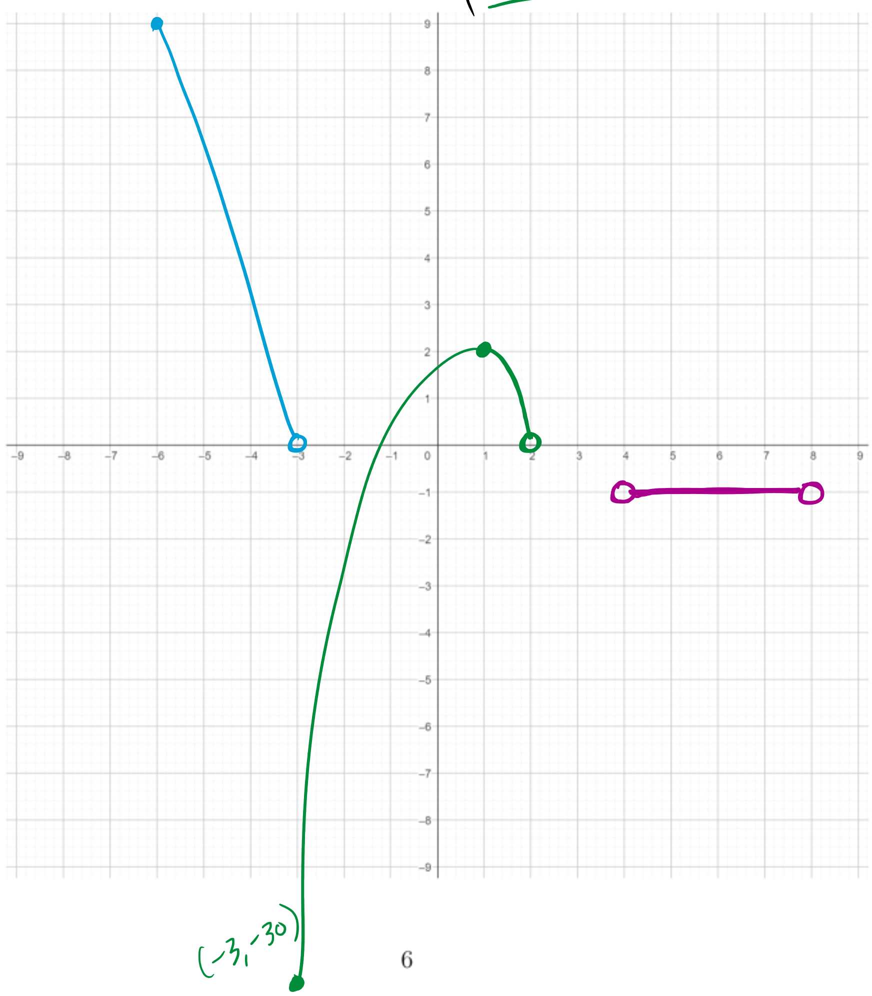
y_2 (as a parabola), still plug-in start & stop x-values, but we also need the vertex.

$y_2(-3) = -2(-3)^2 + 4(-3) = -18 + -12 = -30$ $(-3, -30)$ closed

$y_2(2) = -2(2)^2 + 4(2) = -8 + 8 = 0$ $(2, 0)$ open

vertex: $x = \frac{-b}{2a} = \frac{-4}{2(-2)} = 1$,

$y = y_2(1)$
$= -2(1)^2 + 4(1)$
$= 2$



6

5. Simplify the following exponential expressions. Express you answer without denominators.

(a) $\dfrac{8^{(x+2)}125^{x}}{5^{(-3x+2)}16^{(2x)}}$

← we need to know ~~properties~~ of exponents, in 5.6 notes, there is a box @ the beginning

if you need more help/practice w/ these:

vmlc.tamu.edu

"Workshops" → "Algebra Series"

To be able to combine any of these, we need the bases to match

$8 \to 2^3$, $125 \to 5^3$, $5 \to 5$, $16 \to 2^4$

$$\frac{(2^3)^{x+2} \cdot (5^3)^{x}}{(5)^{-3x+2} \cdot (2^4)^{2x}}$$

$$\frac{a^x}{a^y} \to \boxed{a^{x-y}} \qquad \frac{a^x}{a^y} \to \frac{1}{a^{y-x}}$$

$$(a^x)^y \Rightarrow a^{x \cdot y}$$

$$\frac{2^{3(x+2)} \cdot 5^{3x}}{5^{-3x+2} \cdot 2^{8x}} \to 2^{3(x+2)-8x} \cdot 5^{3x-(-3x+2)} \to 2^{3x+6-8x} \cdot 5^{3x+3x-2}$$

$$\to \boxed{2^{-5x+6} \cdot 5^{6x-2}}$$

(b) $\left(\dfrac{9^{4x}25^{-5y}}{5^{(4-y)}27^{2x}}\right)^{-3}$ ★negative★

★ if there is a power outside, clean up the inside as much as you can first ★

$$\frac{9^{4x} \cdot 25^{-5y}}{5^{4-y} \cdot 27^{2x}} \to \frac{(3^2)^{4x} \cdot (5^2)^{-5y}}{5^{4-y} \cdot (3^3)^{2x}} \to \frac{3^{8x} \cdot 5^{-10y}}{5^{4-y} \cdot 3^{6x}} \to \frac{3^{8x-6x}}{5^{4-y-(-10y)}}$$

$$\to \frac{3^{2x}}{5^{4+9y}} \quad \text{then} \quad \left(\frac{3^{2x}}{5^{4+9y}}\right)^{-3}$$

-3 if this power is negative, then we flip the inside & drop the negative

$$\boxed{\left(\frac{x}{y}\right)^{-2}} \qquad ★\ x^{-a} \to \frac{1}{x^a}$$

$$\frac{1}{\left(\frac{x}{y}\right)^2} \to \frac{1}{\frac{x^2}{y^2}}$$

$$1 \cdot \frac{y^2}{x^2} \to \frac{y^2}{x^2}$$

$$\boxed{(y/x)^2}$$

$$\to \left(\frac{5^{4+9y}}{3^{2x}}\right)^3 \qquad \left(\frac{a}{b}\right)^x \to \frac{a^x}{b^x}$$

$$\to \frac{(5^{4+9y})^3}{(3^{2x})^3} \to \frac{5^{3(4+9y)}}{3^{3(2x)}} \to \frac{5^{12+27y}}{3^{6x}}$$

$$\to \boxed{5^{12+27y} \cdot 3^{-6x}}$$

6. For the following functions, determine the domain and classify them as an exponential function or not. If it is an exponential function, determine if it is exponential growth or exponential decay.

3 restrictions:
① denominators ($\neq 0$)
② even roots (inside ≥ 0)
③ logs (inside > 0)

(a) $f(x) = \dfrac{x^2(x-1)}{x(x-1)(x+8)}$

$x(x-1)(x+8) \neq 0$

$x \neq 0$, $x-1 \neq 0$, $x+8 \neq 0$

$x \neq 1$ $\quad$ $x \neq -8$

D: $(-\infty,-8)\cup(-8,0)\cup(0,1)\cup(1,\infty)$

Not exponential (it is rational)

(b) $g(x) = 3 \cdot (5)^x$ $\quad$ no restrictions

D: $(-\infty,\infty)$ & is exponential

base $= 5 > 1$

exponential growth

(c) $j(x) = \sqrt{19-5x}$ $\quad$ even root:

$19-5x \geq 0$

$-5x \geq -19$

$x \leq 19/5$

D: $(-\infty, 19/5]$ & not exponential

(d) $g(x) = (-6)^{x+2}$ $\quad$ even though we have (number)$^{\text{(variable)}}$

this is Not exponential b/c the base is negative!

no restrictions → D: $(-\infty,\infty)$

(e) $j(x) = (9)^{-4x}$ $\quad$ no restrictions

D: $(-\infty,\infty)$ & it is exponential

If the variable is negative we must be careful!!

$j(x) = \left(\frac{9}{1}\right)^{-4x} = \left(\frac{1}{9}\right)^{4x}$ so base is actually $\frac{1}{9}$!!

since $0 < \frac{1}{9} < 1$ → exponential decay

7. Solve the following equations for x. Express your answer in exact form.

until we get to 5.8, to solve exponential equations, we need bases to match

★ First goal, isolate exponential terms

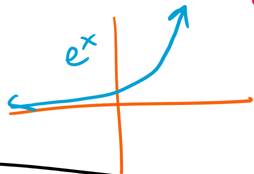
(a) $2e^x + 7 = 0$

only exponential term

$2e^x = -7$

$e^x = -7/2$ ★ red flag

exponential growth function



$\boxed{\text{no solution}}$

★ an exponential expression, if isolated, cannot equal: ① zero ② negative

(b) $\dfrac{1}{6^{-4x}} = 36^{3x-5}$

(a single exponential) = (a single exponential)

we want same base

$6^{-(-4x)} = (6^2)^{3x-5}$

$6^{4x} = 6^{6x-10}$

bases match, drop them

$4x = 6x - 10$

$-2x = -10$

$\boxed{x = 5}$

2 strategies:

① bring up denominator on both sides

② cross-multiply (b/c fraction = fraction)

(c) $\cancel{\dfrac{e^8}{e^{2x}} = \dfrac{e^{7x}}{e^{-2}}}$ $\rightarrow$ $\dfrac{e^8}{e^{2x}} = \dfrac{e^{7x}}{e^{-2}}$

① $\dfrac{e^8}{e^{2x}} = \dfrac{e^{7x}}{e^{-2}}$

$e^{8-2x} = e^{7x-(-2)}$

bases match, drop them

$8 - 2x = 7x + 2$

$6 = 9x$

$x = \frac{6}{9} = \boxed{\frac{2}{3}}$

② $\dfrac{e^8}{e^{2x}} = \dfrac{e^{7x}}{e^{-2}}$

$e^8 \cdot e^{-2} = e^{2x} \cdot e^{7x}$ ★ $a^x \cdot a^y \rightarrow a^{x+y}$

$e^{8+(-2)} = e^{2x+7x}$

bases match, drop

$6 = 9x$

$x = \frac{6}{9} = \boxed{\frac{2}{3}}$

same!

8. You decide to change cellphone providers. Your new provider offers unlimited calls and up to (and including) 25 GB of data for \$50 per month. If you use more than 25 GB of data, then you're charged \$1.50 per additional GB after 25 GB up to (and inlcuding) 40 GB of data. If you use more than 40 GB of data per month, then you're charged \$1.00 per additional GB after 40 GB. Write a piecewise function, $C(d)$, representing the monthly cost in dollars, C, for using d GB of data.

↳ Cost = (cost per item)(quantity) + (fixed cost)

Scenarios:

① Use 0 to 25 Gigs → Cost = (\$0)(d) + (\$50)

② Use more than 25 but not more than 40 → Cost = $(\$1.50)\underbrace{(d-25)}_{\text{each gig after 25}} + (\$50)$

③ Use more than 40 → Cost = $(\$1)\underbrace{(d-40)}_{\text{each gig after 40}} + (\$50) + \underbrace{(\$1.50)(40-25)}_{\text{"new" fixed cost b/c we know we used more than 40 gigs}}$

$$C(d) = \begin{cases} 50, & 0 \le d \le 25 \\ 1.5(d-25) + 50, & 25 < d \le 40 \\ 1(d-40) + 50 + 1.5(15), & d > 40 \end{cases}$$

(depending on prof) might have to simplify

9. You invest \$7,500 in a savings account that earns 4.3% annual interest compounded continuously. How much money will be in the savings account after 10 years if you make no more deposits? Round your answer to the nearest cent. (2 decimals)

$A = Pe^{rt}$ (must know this)

A = future amount of \$
P = initial/starting amount of \$
r = interest rate (as a decimal)
t = time (in years)

A = ?
P = 7500
r = 0.043
t = 10

$A = 7500e^{(0.043)(10)}$

7500e^(0.043 • 10)

A = 11529.43143-----

$\boxed{\text{the amount in the account after 10 years is } \$11,529.43}$