2024 Fall Math 140 Week-In-Review

Week 9: Sections 5.3 and 5.4

Some Key Words and Terms: Domain, Rational Functions, Holes, Vertical Asymptotes, Intercepts, Simplifying Rational Expressions, Difference Quotient, Power/Radical Functions, Converting Exponents, Conjugate, Rationalizing.

Domain: 3 domain restrictions:

1. denominator $(\neq 0)$
2. even root (inside ≥ 0)
3. logs (inside > 0)

Rational Function:

★ $\frac{\text{polynomial}}{\text{polynomial}}$ ★ b/c this is a fraction (so has a denominator) we will always check for domain restrictions

- domain restrictions for rational function can only be: a hole or VA
- to determine, we factor & reduce the rational function

Holes: an x-value is a hole if the term in the denominator completely cancels that gave that x-value:

$$f(x) = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x-2)} \rightarrow \frac{x+1}{x-2}$$

$x \neq 1 \quad x \neq 2$

$(x-1)$ completely cancelled from the bottom, so $x=1$ is where a hole is at.

Vertical Asymptotes: an x-value is a VA if the term in the denominator does not completely cancel that gave that x-value

$$f(x) = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x-2)} \rightarrow \frac{x+1}{x-2}$$

$x \neq 1 \quad x \neq 2$

$(x-2)$ did not cancel from the bottom, so $x=2$ is where a VA is at.

Intercepts:

y-int: plug-in $x=0$ to function

x-int: w/ any function we set the whole function $= 0$ $\left(\frac{\text{top}}{\text{bottom}} = 0\right) \cdot \text{bottom}$

$\text{top} = (0) \cdot (\text{bottom})$

$\text{top} = 0$ ★ always check w/ domain

Simplifying Rational Expressions: adding/subtracting/multiplying/dividing fractions

add/subtract: ① factoring & ② common denominators

multiply/divide: ① factoring & ② converting division $\rightarrow$ multiplication

Difference Quotient: ☆ $\frac{f(x+h)-f(x)}{h}$ ☆ be able to use with:

1) polynomials (multiplying everything out & combine terms)
2) rationals (will always a common denominator)
3) radicals (will always use the conjugate of the top)

Power/Radical Functions:

Power: $(\text{variable})^{(\text{number})}$ ← can be any number: decimal, fraction, +/–, ...

Radical: $\sqrt[n]{(\text{inside})}$ n is the "index": 1) odd (no restrictions) 2) even (check for restrictions)

Converting Exponents:

☆ $x^{a/b} \longleftrightarrow \sqrt[b]{x^a}$

In general: $\underbrace{(\text{expression})^{a/b}}_{\text{better for derivatives (Math 142)}} \longleftrightarrow \underbrace{\sqrt[b]{(\text{expression})^a}}_{\text{better for domain}}$

Conjugate:

$x+2$ & $x-2$ are conjugates
$-x+1$ & $-x-1$ are conjugates
$5-x$ & $5+x$ are conjugates

$\underbrace{(a)+(b) \longleftrightarrow (a)-(b)}$

☆ $\underbrace{(a+b)(a-b) = a^2-b^2}_{\text{helpful to remember}}$ ☆

Rationalizing: ☆ you are most likely to see this with difference quotients & radical function ☆

we multiply the fraction (if not put over 1) top & bottom by the conjugate of the radical term

$$\frac{3x}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \text{------}$$

★ for help on factoring, go to $\boxed{\text{vmlc.tamu.edu}}$ & Workshops → Algebra Series "solving quadratic equations"

Examples:

1. Determine the domain, in interval notation, for the following functions.

(a) $f(x) = \dfrac{-x^3 + 5x^2}{x^2 - 7x + 10}$

denominator:

$x^2 - 7x + 10 \neq 0$

$(x-5)(x-2) \neq 0$

$x - 5 \neq 0$, $x - 2 \neq 0$

$x \neq 5$, $x \neq 2$

2 5

$\boxed{D: (-\infty, 2) \cup (2, 5) \cup (5, \infty)}$

(b) $g(x) = 8\sqrt[3]{5x+2} + 3\sqrt[8]{2x+5}$

odd root: no restrictions

even root:

$2x + 5 \geq 0$

$2x \geq -5$

$x \geq -5/2$

$-5/2$

$\boxed{D: \left[-\frac{5}{2}, \infty\right)}$

(c) $h(x) = \dfrac{2\sqrt{9-8x}}{x^2 - 5}$

denominator:

$x^2 - 5 \neq 0$

$x^2 \neq 5$

$\sqrt{x^2} \neq \pm\sqrt{5}$

$x \neq \pm\sqrt{5}$

★ x^2 = negative
no solution

even root:

$9 - 8x \geq 0$

$\dfrac{-8x}{-8} \geq \dfrac{-9}{-8}$

$x \leq 9/8$

★ mult/div by a negative, flip the inequality

overlap (what is in common)

$-\sqrt{5}$ $9/8$ $\sqrt{5}$

$\boxed{D: (-\infty, -\sqrt{5}) \cup (-\sqrt{5}, 9/8]}$

(d) $j(x) = \dfrac{5x^2 + 6x - 9}{\sqrt[4]{3x-7}}$

even root: $3x - 7 \geq 0$

denominator: $\sqrt[4]{3x-7} \neq 0$

★ the denominator is an even root ★

$3x - 7 > 0$

$3x > 7$

$x > 7/3$

$7/3$

$\boxed{D: (7/3, \infty)}$

2. For the functions given, determine all intercepts and any holes or vertical asymptotes for the function.

rational functions
★ start w/ domain ★

(a) $f(x) = \dfrac{-x^3+5x^2}{x^2-7x+10} = \dfrac{-x^2\cancel{(x-5)}}{\cancel{(x-5)}(x-2)}$

$x \neq 5$ (hole) $\quad x \neq 2$ (VA)

canceled

can't use same name

$g(x) = \dfrac{-x^2}{x-2}$

$y = \dfrac{-(5)^2}{5-2} = \dfrac{-25}{3}$

y-int: $\dfrac{-(0)^2}{0-2} = \dfrac{0}{-2} = 0$

x-int: $\dfrac{-x^2}{x-2} = 0 \rightarrow -x^2 = 0 \rightarrow x = 0$

$\boxed{D: (-\infty, 2) \cup (2,5) \cup (5, \infty)}$

★ after domain, use the reduced version for everything else ★

$\boxed{VA: \; x = 2}$ (written as vertical line)

$\boxed{Hole: \; (5, -25/3)}$ (written as a point)

$\boxed{\text{Yint: } (0,0) \quad \text{Xint: } (0,0)}$

(b) $h(x) = \dfrac{(x-3)(x+1)^{\cancel{2}}(x+5)}{x\cancel{(x+1)}(x-5)^{\cancel{3}\,2}}$

should be a minus

$x \neq 0$, $x \neq -1$, $x \neq 5$

$g(x) = \dfrac{(x-3)(x+1)}{x(x-5)^2}$ so

$y = \dfrac{(-1-3)(-1+1)}{(-1)(-1-5)^2} = \dfrac{0}{-36} = 0$ (for hole)

y-int: none b/c $x \neq 0$

x-int: $\dfrac{(x-3)(x+1)}{x(x-5)^2} = 0 \rightarrow (x-3)(x+1) = 0$

$x = 3$, ~~$x = -1$~~

D: $(-\infty, -1) \cup (-1, 0) \cup (0, 5) \cup (5, \infty)$

VA: $x = 0$ & $x = 5$

Hole: $(-1, 0)$

Y-int: None

X-int: $(3, 0)$

3. Simplify the following rational expressions. Express your answer using only positive exponents.

(a) $\dfrac{x+3}{(x-2)} - \dfrac{2x}{(x-1)}$

subtracting, so common denominator

$(x-2)(x-1)$ this a common denominator

$\dfrac{(x+3)}{(x-2)} \cdot \dfrac{(x-1)}{(x-1)} - \dfrac{(2x)}{(x-1)} \cdot \dfrac{(x-2)}{(x-2)}$

must equal 1 $\qquad$ must equal 1

★ generally speaking, keep bottom factored

$(x+3)(x-1)$
$x^2 - x + 3x - 3$
$x^2 + 2x - 3$

$2x(x-2)$
$2x^2 - 4x$

$\dfrac{(x^2+2x-3) - (2x^2-4x)}{(x-2)(x-1)}$

$\dfrac{x^2+2x-3 - 2x^2+4x}{(x-2)(x-1)}$

$\boxed{\dfrac{-x^2+6x-3}{(x-2)(x-1)}}$

top doesn't factor, so done

mult/div → factor

(b) $\dfrac{x^3-9x}{\cancel{x-6x+8}} \div \dfrac{x^2-x-6}{2x^2-8}$

x^2-6x+8

$$\frac{x(x^2-9)}{(x-2)(x-4)} \div \frac{(x-3)(x+2)}{2(x^2-4)}$$

$$\frac{x(x-3)(x+3)}{(x-2)(x-4)} \div \frac{(x-3)\cancel{(x+2)}}{2\cancel{(x+2)}(x-2)}$$

★ now, reduce within each fraction ★

$$\frac{x(x-3)(x+3)}{(x-2)(x-4)} \div \frac{(x-3)}{2(x-2)}$$

★ now, flip the fraction that is doing the dividing

$$\frac{x\cancel{(x-3)}(x+3)}{\cancel{(x-2)}(x-4)} \cdot \frac{2\cancel{(x-2)}}{\cancel{(x-3)}}$$

since multiplication, we can cancel between any top & bottom

$$\boxed{\frac{2x(x+3)}{x-4}}$$

(c) $\left(\dfrac{\cancel{(2xyz)^3}}{22x^{-2}y^5z}\right)^{-2}$ $(2xyz)^3$

$(\text{fraction})^{\text{power}}$ generally best to simplify inside first

$(2xyz)^3 \rightarrow (xy)^a \rightarrow x^ay^a \rightarrow 2^3x^3y^3z^3 \rightarrow 8x^3y^3z^3$

$$\frac{{}^{4}\cancel{8}x^3y^3z^3}{{}_{11}\cancel{22}x^{-2}y^5z^1} \rightarrow \frac{x^a}{x^b} \begin{cases} \nearrow x^{a-b} \\ \searrow \dfrac{1}{x^{b-a}} \end{cases}$$

to keep power positive, move the smaller power to the larger power

$(2xyz)^3 = (2xyz)(2xyz)(2xyz)$

$= \underbrace{2\cdot2\cdot2}\cdot\underbrace{x\cdot x\cdot x}\cdot\underbrace{y\cdot y\cdot y}\cdot\underbrace{z\cdot z\cdot z}$

$8x^3y^3z^3$

$$\frac{4x^{3-(-2)}z^{3-(1)}}{11y^{5-(3)}} \rightarrow \frac{4x^5z^2}{11y^2}$$

(simplified inside)

$$\left(\frac{4x^5z^2}{11y^2}\right)^{-2}$$

to get rid of negative, flip inside

$$\rightarrow \left(\frac{(11)(y^2)}{(4)(x^5)(z^2)}\right)^2$$

★ $(xy)^a \rightarrow x^ay^a$

★ $\left(\frac{x}{y}\right)^b \rightarrow \frac{x^b}{y^b}$

$$\frac{(11)^2(y^2)^2}{(4)^2(x^5)^2(z^2)^2} \rightarrow \boxed{\frac{121y^4}{16x^{10}z^4}}$$

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★ $(x^a)^b \rightarrow x^{a\cdot b}$

4. For the following functions, convert from radical form to power form. Express your answer without denominators.

$$\sqrt[b]{x^a} \to x^{a/b}$$

(a) $f(x) = 4\sqrt[3]{(x^2-6x)^7}$ coefficient rewrite

$$\boxed{f(x) = 4(x^2-6x)^{7/3}}$$

(b) $g(x) = \dfrac{7}{\sqrt[6]{(8-3x)^5}} \to \dfrac{7}{(8-3x)^{5/6}}$

$$= \boxed{7(8-3x)^{-5/6}}$$

☆ $\dfrac{1}{x^{-a}} \to x^a$

$\dfrac{1}{x^a} \to x^{-a}$

5. For the following functions, convert from power form to radical form. Express your answer without negative exponents.

$$x^{a/b} \to \sqrt[b]{x^a}$$

(a) $f(x) = x^{1/2} - 3x^{-1/2} + (7x)^{-5/4}$

3 does not move (power only to x)

7 does move (power to both)

$$f(x) = x^{1/2} - \frac{3}{x^{1/2}} + \frac{1}{(7x)^{5/4}}$$

$$f(x) = \sqrt[2]{x^1} - \frac{3}{\sqrt[2]{x^1}} + \frac{1}{\sqrt[4]{(7x)^5}}$$

OR $\boxed{f(x) = \sqrt{x} - \dfrac{3}{\sqrt{x}} + \dfrac{1}{\sqrt[4]{(7x)^5}}}$

☆ with negative exponents, we make the positive before writing as radicals ☆

(b) $g(x) = x^{3/4}(x^2+2)^{-3/2}$

$$g(x) = \frac{x^{3/4}}{(x^2+2)^{3/2}} = \boxed{\frac{\sqrt[4]{x^3}}{\sqrt{(x^2+2)^3}}}$$

6. For the following expressions, determine the conjugate.

$a-b \to a+b$

$a+b \to a-b$

(a) $x - 5 \longrightarrow x+5$

$a=x \quad b=5$

(b) $2x - \sqrt{x} \longrightarrow 2x+\sqrt{x}$

$a=2x \quad b=\sqrt{x}$

(c) $\sqrt{x+3} + \sqrt{11} \longrightarrow \sqrt{x+3} - \sqrt{11}$

$a=\sqrt{x+3} \quad b=\sqrt{11}$

$(x-5)(x+5) \to (x)^2 - (5)^2$

$(2x-\sqrt{x})(2x+\sqrt{x}) \to (2x)^2 - (\sqrt{x})^2$

$4x^2 - x$

7. For the given functions, setup and fully simplifly the difference quotient.

$$\frac{f(x+h)-f(x)}{h}$$

(a) $f(x) = 2x^2 - 5x$ (polynomial, multiply out & combine terms)

1, 2, 3

$\star$ $(x+h)^2 = (x+h)(x+h) = x^2+2xh+h^2$ $\star$

① $f(x+h) = 2(x+h)^2 - 5(x+h)$

$= 2(x^2+2xh+h^2) - 5(x+h)$

$= 2x^2+4xh+2h^2-5x-5h$ (not uncommon to have no like terms)

② $f(x+h) - f(x) = (2x^2+4xh+2h^2-5x-5h) - (2x^2-5x)$

$= \cancel{2x^2}+4xh+2h^2\cancel{-5x}-5h\cancel{-2x^2}\cancel{+5x}$

$= 4xh+2h^2-5h$ (everything left here should have an "h")

③ $\dfrac{f(x+h)-f(x)}{h} = \dfrac{4xh+2h^2-5h}{h}$

$\star$ last step is to factor h out of the top & cancel w/ the h on bottom

$= \dfrac{\cancel{h}(4x+2h-5)}{\cancel{h}} = \boxed{4x+2h-5}$

w/ difference quotient, once you cancel the h the started on bottom, you're done

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$\dfrac{f(x+h)-f(x)}{h}$ w/ rational → common denominator

(b) $g(x) = \dfrac{x+2}{\ldots}$

$\frac{f(x+h)-f(x)}{h}$ w/ rational $\to$ common denominator

(b) $g(x) = \frac{x+2}{x}$

① $g(x+h) = \frac{(x+h)+2}{(x+h)} = \frac{x+h+2}{x+h}$

subtract fractions: $(x+h)(x)$ as common denominator

② $g(x+h) - g(x) = \frac{x+h+2}{x+h} - \frac{x+2}{x}$

$\frac{x+h+2}{x+h} \cdot \frac{x}{x} - \frac{x+2}{x} \cdot \frac{x+h}{x+h} \to \frac{x(x+h+2) - \overbrace{(x+2)(x+h)}^{\text{FOIL}}}{x(x+h)}$

$\to \frac{x^2+xh+2x - (x^2+xh+2x+2h)}{x(x+h)} = \frac{\cancel{x^2}+\cancel{xh}+\cancel{2x}-\cancel{x^2}-\cancel{xh}-\cancel{2x}-2h}{x(x+h)}$

③ $\frac{g(x+h)-g(x)}{h} = \frac{\frac{-2h}{x(x+h)}}{\frac{h}{1}} = \frac{-2\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \boxed{\frac{-2}{x(x+h)}}$

Faster ---

$g(x+h) = \frac{x+h+2}{x}$

☆ still going to use $x(x+h)$ common denominator

$\frac{g(x+h)-g(x)}{h} = \frac{\left(\frac{x+h+2}{x+h} - \frac{x+2}{x}\right)}{h} \cdot \frac{x(x+h)}{x(x+h)}$

$= \frac{\frac{(x+h+2)(x)\cancel{(x+h)}}{\cancel{x+h}} - \frac{(x+2)\cancel{(x)}(x+h)}{\cancel{x}}}{h \cdot x(x+h)} = \frac{(x+h+2)(x) - (x+2)(x+h)}{h \cdot x(x+h)}$

exact same as other method

$= \frac{x^2+xh+2x - (x^2+xh+2x+2h)}{h \cdot x(x+h)}$

$= \frac{\cancel{x^2}+\cancel{xh}+\cancel{2x}-\cancel{x^2}-\cancel{xh}-\cancel{2x}-2h}{h \cdot x(x+h)} = \frac{-2\cancel{h}}{\cancel{h} \cdot x(x+h)}$

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$= \frac{-2}{x(x+h)}$

$$\frac{f(x+h)-f(x)}{h}$$

(c) $j(x)=\sqrt{2x-1}$

radical, the 1, 2, 3 method doesn't work b/c nothing simplifies...

① $j(x+h)=\sqrt{2(x+h)-1}=\sqrt{2x+2h-1}$

② $j(x+h)-j(x)=\sqrt{2x+2h-1}-\sqrt{2x-1}$

☆ cannot combine these b/c not exactly the same under the radical

③ $$\frac{j(x+h)-j(x)}{h}=\frac{\sqrt{2x+2h-1}-\sqrt{2x-1}}{h}\cdot\frac{\sqrt{2x+2h-1}+\sqrt{2x-1}}{\sqrt{2x+2h-1}+\sqrt{2x-1}}$$

must be in parenthesis

$$=\frac{\left(\sqrt{2x+2h-1}\right)^2-\left(\sqrt{2x-1}\right)^2}{h\cdot\left[\sqrt{2x+2h-1}+\sqrt{2x-1}\right]}=\frac{(2x+2h-1)-(2x-1)}{h\left(\sqrt{2x+2h-1}+\sqrt{2x-1}\right)}$$

$$=\frac{\cancel{2x}+2h-\cancel{1}-\cancel{2x}+\cancel{1}}{h\left(\sqrt{2x+2h-1}+\sqrt{2x-1}\right)}=\frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2x+2h-1}+\sqrt{2x-1}\right)}$$

cancelling this is the last step

$$=\boxed{\frac{2}{\sqrt{2x+2h-1}+\sqrt{2x-1}}}$$