Problem 1. Find two positive numbers such that the sum of the first and thrice the second number is 60 and the product of the numbers is a maxima.

Ans: The numbers are $x=30$ and $y=10$
max product $= (30)(10) = 300$

Let the numbers be x and y.

$x + 3y = 60 \rightarrow$ constraint

rewrite
$3y = 60 - x$
$y = \frac{60-x}{3}$

$\boxed{P = (x)(y)} \leftarrow$ optimize this value $\rightarrow$ maxima. $\downarrow$ CP $\downarrow$ $f'(x) = 0$ / DNE

$P = (x)\cdot\left(\frac{60-x}{3}\right) = \frac{60x - x^2}{3} = \frac{60x}{3} - \frac{x^2}{3}$

$P = 20x - \frac{x^2}{3}$

$\frac{d}{dx}(P) = \frac{d}{dx}\left(20x - \frac{x^2}{3}\right) = 20 - \frac{2x}{3} = 0$ to find cp.

$20 - \frac{2x}{3} = 0 \Rightarrow 20 = \frac{2x}{3} \Rightarrow x = \frac{20(3)}{2} = 30$ | $y = \frac{60-x}{3} = \frac{60-30}{3} = \frac{30}{3} = 10$

Problem 2. The sum of 2 positive numbers is 16. What is the smallest possible value of the sum of their squares? Show that this value is a minimum by using second derivative test.

Let my numbers be x and y.

constraint : $x + y = 16 \longrightarrow x = 16 - y$

optimize: $x^2 + y^2 = S$

$(16-y)^2 + y^2 = S$

$[(16)^2 - 2(16)(y) + y^2] + y^2 = S$

$(16)^2 - 32y + 2y^2 = S$

$(16-y)^2 = (16-y)(16-y)$
$= 16(16-y) - y(16-y)$
$= (16)(16) - 16y - 16y + y^2$

Find cp: $\frac{d}{dy}(S) = S' = 0 - 32 + 4y = 0$

$-32 + 4y = 0$
$4y = 32$
$y = 8$

$x = 16 - y$
$= 16 - 8$
$= 8$

find 2nd derivative of S

$\frac{d}{dy}(S') = S''$

$\frac{d}{dy}(-32 + 4y) =$
$0 + 4$
$4 > 0$
$\therefore$ CP must be a minima.

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

Ans: Smallest sum of squares is $(8)^2 + (8)^2 = 64 + 64 = 128$

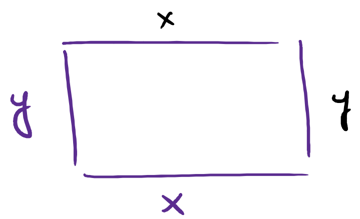
Ans: The recangle with smallest perimetn has sides
$x = 15\text{cm}$ and $y = 15\text{ cm}$

2

Smallest Perimeter $= 2(15) + 2(15) = 60\text{cm}$ $\quad xy$

Problem 3. Find the dimensions of a rectangle which has an area of 225 square centimeters and the smallest possible perimeter. What is the perimeter of this rectangle?

Let the rectangle have sides x and y.



Constraint: $\quad xy = 225 \Rightarrow y = \dfrac{225}{x}$

$y = \dfrac{225}{15} = 15$

Optimize: $\quad P = 2x + 2y$

$= 2x + 2 \cdot \left(\dfrac{225}{x}\right)$

$= 2x + \dfrac{450}{x}$

Recap:

$\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$

$= \dfrac{d}{dx}(x^{-1})$

$= (-1)x^{-2}$

$= -\dfrac{1}{x^2}$

Look for CP: $\quad P' = 0$

$P' = 2 + (450)\left(-\dfrac{1}{x^2}\right) = 0$

$2 - \dfrac{450}{x^2} = 0$

$\dfrac{450}{x^2} = 2$

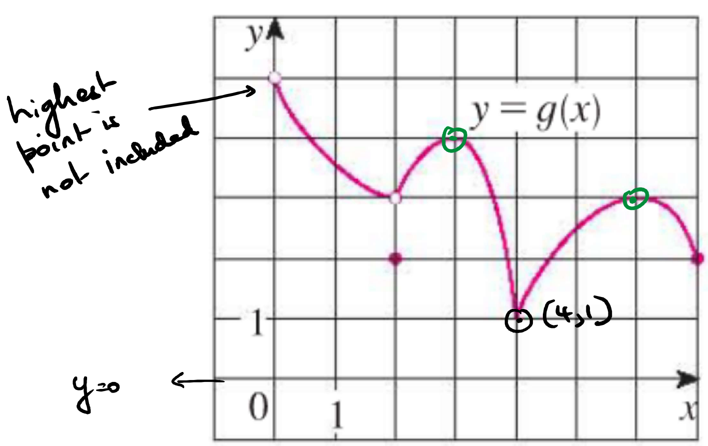
$2x^2 = 450$

$x^2 = \dfrac{450}{2} = 225$

$x = \sqrt{225} = 15$

Problem 4. What is the absolute maxima and the absolute minima of the function given below? Are there any local extremas?

D: $(0, 7]$



① Absolute minima on D is $y = 1$ at $x = 4$

② Absolute maxima: DNE

③ Local maxima @ $x = 3$ & $x = 6$

④ Local minima @ $x = 4$

$\frac{d}{dx}(x^2) = 2x$

Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

$a^2 - b^2 = (a+b)(a-b)$

Problem 5. Find the absolute maxima and the absolute minima for $f(x) = x^4 - 18x^2 + 32$ on the interval $[-4, 4]$.

$f(x) = x^4 - 18x^2 + 32$ is a continuous function

Find CPs : $f'(x) = 0$ & solve for x

$f'(x) = 4x^3 - 18(2x) + 0$

Interval : $[-4, 4]$

$36x = (4)(9)(x)$

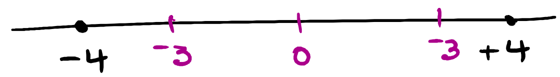
$\frac{36x}{4x} = 9$

$4x^3 - 36x = 0$

$4x(x^2 - 9) = 0$.

$4x(x+3)(x-3) = 0$

$x = 0$ $\quad x = -3$ $\quad x = 3$



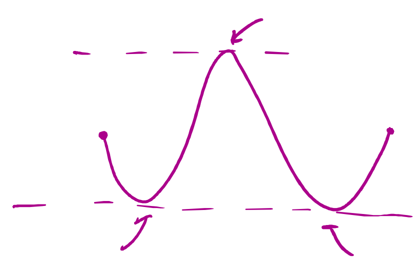
$f(x) = x^4 - 18x^2 + 32$.

end points
① $f(-4) = 0$
② $f(4) = 0$

CPs
③ $f(0) = 32$
④ $f(-3) = -49$
⑤ $f(+3) = -49$

Absolute maxima is $y = 32$ at $x = 0$.

Absolute minima is $y = -49$ at $x = \pm 3$



Ans: The box must have $l = 4m$, $w = 4m$ and $h = 2m$ in order to have the largest possible volume.

no top.
$\downarrow$

$l = w = x$

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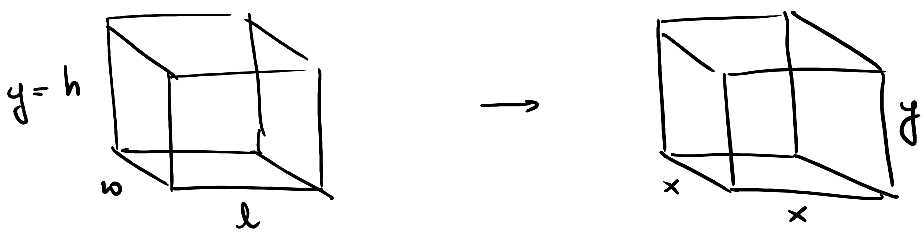
Problem 6. You would like to make an open rectangular box with a square base from 48 m^2 of material. Find the dimensions of the box that will result in it's largest possible volume.



Constraint : surface area $= 48\,m^2$

$$\underbrace{\left(x^2\right)}_{\text{floor}} + \underbrace{\left(xy + xy + xy + xy\right)}_{\text{4 walls.}} = 48$$

$$\boxed{x^2 + 4xy = 48}$$

$$\frac{4xy}{4x} = \frac{48 - x^2}{4x} \Bigg\} \quad y = \frac{48}{4x} - \frac{x^2}{4x}$$

$$y = \frac{12}{x} - \frac{x}{4}$$

Optimize.

Volume $V = (x)(x)(y) = x^2 y$

$$V = x^2\left(\frac{12}{x} - \frac{x}{4}\right) = \frac{12x^2}{x} - \frac{x^3}{4} = 12x - \frac{x^3}{4}$$

For CP : $V' = 0$

$$V = 12x - \frac{x^3}{4}$$

$$V' = 12 - \frac{1}{4}\left(3x^2\right)$$

$$12 - \frac{3x^2}{4} = 0$$

$$12 = \frac{3x^2}{4}$$

$$12(4) = 3x^2$$

$$\frac{\cancel{12}^{\,4}(4)}{\cancel{3}} = x^2$$

$$x^2 = 16$$

Ans.

$$x = \sqrt{16} = 4$$

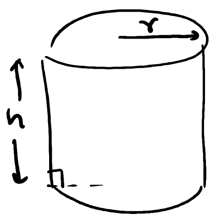
$$y = \frac{12}{x} - \frac{x}{4} = \frac{12}{4} - \frac{4}{4} = 3 - 1 = 2$$

$\frac{\ }{4} = 0$ $\quad x^2 = 16 \quad = 3-1 = 2$

Ans: The cylinder with largest volume will have $r = 1.77\,m$ and $h = 1.77\,m$.

Problem 7. A container, in the shape of a right circular cylinder with no top, has surface area of 3 m^2. Find the height h and base radius r that will maximize the volume of this cylinder. The volume of a cylinder is $\pi r^2 h$.

optimize



formula.

$V = \pi r^2 h$

Surface area $= \pi r^2 + (2\pi r) h \quad + \pi r^2$ → no top.

constraint ← Surface area $= \boxed{\pi r^2 + 2\pi r h} = 3\ m^2$

$2\pi r h = 3 - \pi r^2$

$h = \frac{3 - \pi r^2}{2\pi r}$

$V = \pi r^2 h = \cancel{\pi} r^{\cancel{2}} \left(\frac{3 - \pi r^2}{2 \cancel{\pi} \cancel{r}} \right)$

$= \frac{r(3 - \pi r^2)}{2} = \frac{3r - \pi r^3}{2} = \frac{3r}{2} - \frac{\pi r^3}{2}$

$V = \frac{3}{2} r - \frac{\pi}{2} \cdot r^3$

$\frac{d}{dr}(v) = V' = \underbrace{\frac{3}{2}(1) - \frac{\pi}{2}(3r^2)}_{V' = 0} = 0$

$\frac{3}{2} - \frac{3\pi}{2} r^2 = 0$

solve for r

$\frac{3}{2} = \frac{3\pi}{2} \cdot r^2$

$r^2 = \left(\frac{3}{3}\right) \ldots$

$r^2 = \frac{1}{\pi}$

$r = \frac{1}{\sqrt{\pi}} = 1.77$

$h = \frac{3 - \pi r^2}{2\pi r} = 1.77$

$= \cancel{\frac{3 - \pi \cdot \left(\frac{1}{\pi}\right)}{2\pi \left(\frac{1}{\sqrt{\pi}}\right)}}$

$= \frac{3-1}{\ldots} = 2 \ldots$

$$r^2 = \frac{\left(\frac{3}{2}\right)}{\left(\frac{3}{2}\pi\right)} = \frac{1}{\pi}$$

$$2\pi\left(\frac{1}{\sqrt{\pi}}\right) = \frac{3-1}{2\sqrt{\pi}} = \frac{2}{2\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}$$

Problem 8. Find the absolute maxima and the absolute minima of the function

$f(x) = \frac{1}{3}x^3 + 2x^2 - 21x + 7$ on the interval $[0, 6]$.

Let's find any CP between $[0, 6]$

check for CP within interval

$$f(x) = \frac{1}{3}x^3 + 2x^2 - 21x + 7$$

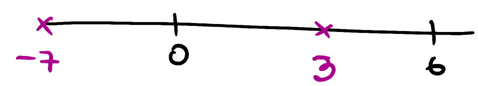
$$f'(x) = \frac{1}{\cancel{3}}(\cancel{3}x^2) + 2\cdot(2x) - 21(1) + 0$$

$$f'(x) = x^2 + 4x - 21 = 0$$

$$(x+7)(x-3) = 0$$

$$\Downarrow \qquad \Downarrow$$

$$\cancel{x = -7} \qquad x = +3$$



only CP

Test points on $f(x) = \frac{1}{3}x^3 + 2x^2 - 21x + 7$

end points
① $f(0) = 7$
② $f(6) = 25$

CP ← ③ $f(3) = -29$

Ans: Absolute maxima is $y = 25$ at $x = 6$

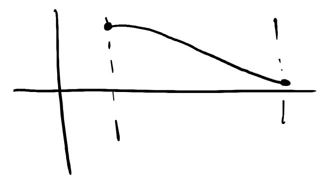
Absolute minima is $y = -29$ at $x = 3$

Problem 9. A rectangular garden needs to be fenced. There is \$320 available for this project. Three sides of the fence will be constructed with wire fencing at the cost of \$2 per foot. The fourth side will be constructed with wood fencing at a cost of \$6 per foot. Find the length of the sides as well as the area of the largest garden that can be fenced in this way.

Problem 10. If a function $f(x)$ is continuous on an interval $[a, b]$, discuss whether the following statements are true of false.

Ex

(1) $f(x)$ must have a local maxima and a local minima on $[a, b]$.



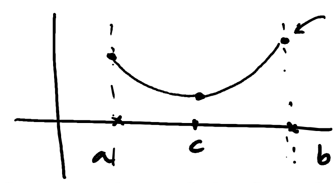
FALSE

(2) $f(x)$ must have an absolute maxima and an absolute minima on $[a, b]$.

TRUE

local max/min

(3) If $x = c$ is the only critical point on the interval $[a, b]$, and $f(c)$ is a local minima, then $f(x)$ has an absolute minima at $x = c$.



TRUE

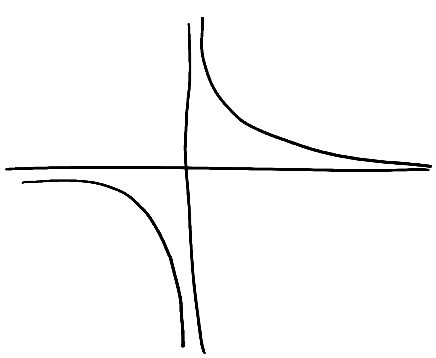
(4) The point $x = b$ can be a local maxima as well as an absolute maxima of the function $f(x)$.

end points cant be local max/min

FALSE

Problem 11. Does the function $f(x) = \dfrac{1}{x}$ have an absolute maxima or an absolute minima on the interval $(-\infty, \infty)$? What about any local extema?

$f(x) = \dfrac{1}{x}$



as $x \to 0^-$, $f(x) \to -\infty$
as $x \to 0^+$, $f(x) \to +\infty$ $\Big\}$ vertical asymptote

$f(x)$ does not have an absolute maxima or minima.

Also, no local extrema

as $x \to \pm\infty$, $f(x) \to 0$

$$\frac{d}{dx}\left(\frac{hi}{lo}\right) = \frac{lo\,\frac{d}{dx}(hi) - hi\,\frac{d}{dx}(lo)}{lo \cdot lo}$$

quotient rule

Problem 12. Find the absolute extremas for $f(x) = -\frac{x+4}{x-4}$ on the interval $[0, 3]$.

$$f(x) = -\frac{x+4}{x-4} = \frac{-x-4}{x-4}$$

$$f'(x) = \frac{(x-4)(-1) - (-x-4)(1)}{(x-4)^2} = \frac{-x+4+x+4}{(x-4)^2} = \frac{8}{(x-4)^2}$$

CP's : $f'(x) = 0$ @ no value of x → no CPs.

$f'(x)$ DNE @ $x = 4$ → outside $[0,3]$ & not in Domain

end points

① $f(0) = -\left(\frac{0+4}{0-4}\right) = -\frac{4}{-4} = 1$ ← Absoute minima @ $x=0$

② $f(3) = -\left(\frac{3+4}{3-4}\right) = -\frac{7}{-1} = 7$ ← Absolute maxima @ $x = 3$

Problem 13. Find the absolute extremas for $f(x) = \frac{\ln x}{x}$ on the interval $[1, 4]$.

$$f(x) = \frac{\ln(x)}{x}$$

Find CPs.

$$f'(x) = \frac{x \cdot \left(\frac{1}{x}\right) - \ln(x) \cdot (1)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

$f'(x) = 0 \Rightarrow 1 - \ln(x) = 0 \Rightarrow 1 = \ln(x)$

$x = e^1 = e$

$f'(x)$ DNE at $x = 0$ → not in Domain

end points

① $f(1) = \frac{\ln(1)}{1} = 0$ ← Absolute minima @ $x=1$

② $f(4) = \frac{\ln(4)}{4} = 0.3465$

CP. ③ $f(e) = \frac{\ln(e)}{e} = \frac{1}{e} = 0.3678$ ← Absolute maxima @ $x = e$

$x = 2.1718$