Week in Review
Math 152

Week 02

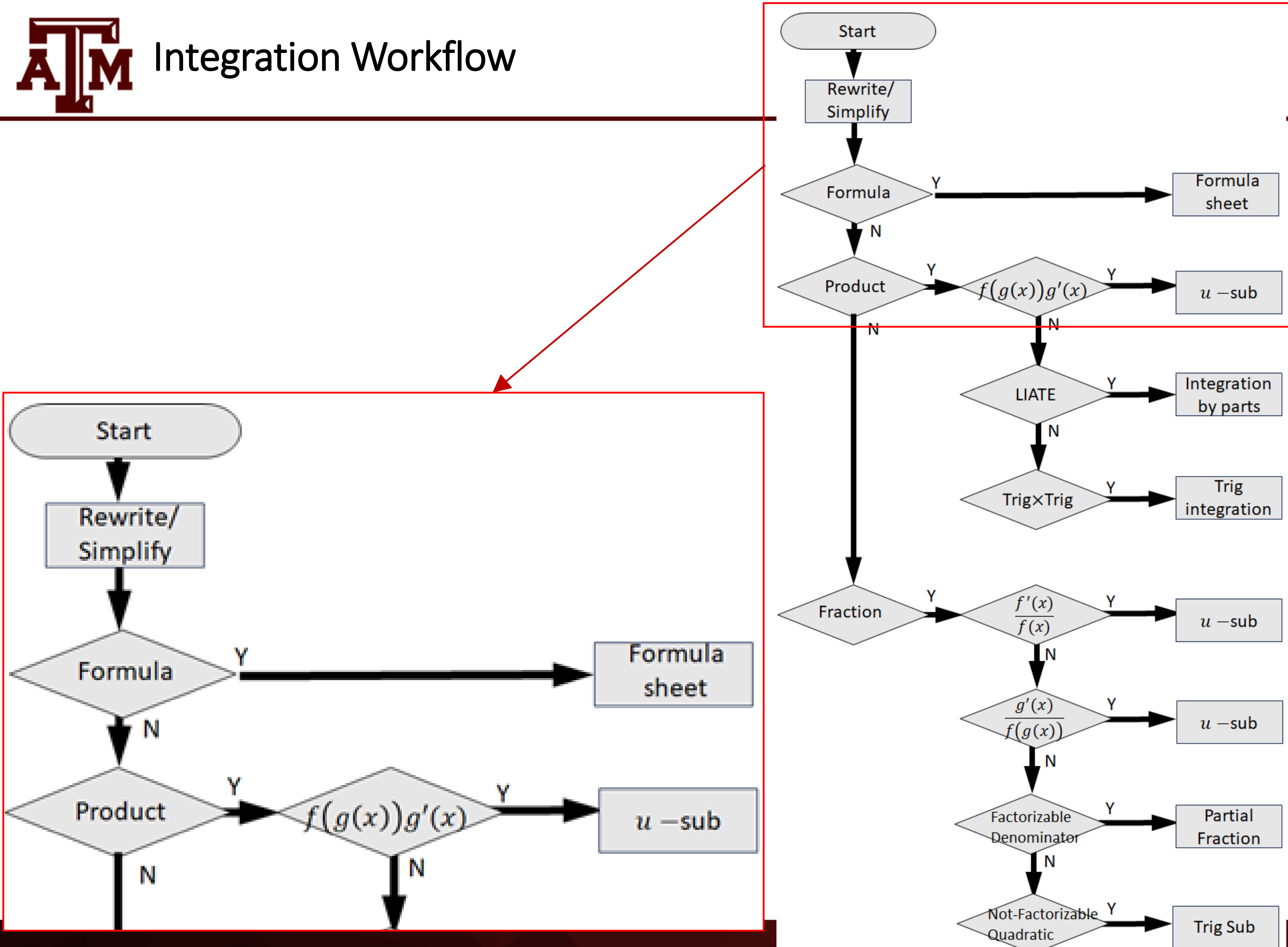
The Substitution Rule

Area Between Curves

Integration formulas

$\int 0\, dx =$	C
$\int dx =$	$x + C$
$\int x^r dx = \quad (r \neq -1)$	$\frac{x^{r+1}}{r+1} + C$
$\int \cos x\, dx =$	$\sin x + C$
$\int \sin x\, dx =$	$-\cos x + C$
$\int \sec^2 x\, dx =$	$\tan x + C$
$\int \sec x \tan x\, dx =$	$\sec x + C$
$\int \csc x \cot x\, dx =$	$-\csc x + C$
$\int e^x dx =$	$e^x + C$
$\int b^x dx = \ (0 < b, b \neq 1)$	$\frac{b^x}{\ln b} + C$
$\int \frac{1}{x} dx =$	$\ln\lvert x\rvert + C$
$\int \frac{1}{1+x^2} dx =$	$\tan^{-1} x + C$
$\int \frac{1}{\sqrt{1-x^2}} dx =$	$\sin^{-1} x + C$
$\int \frac{1}{x\sqrt{x^2-1}} dx =$	$\sec^{-1}\lvert x\rvert + C$

Integration Workflow

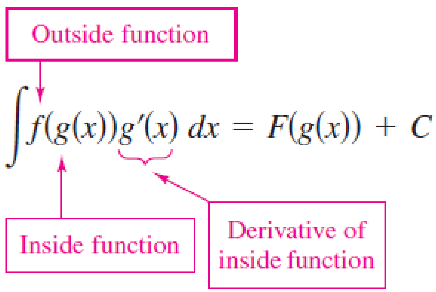


$\boldsymbol{u}$ −substitution to find an antiderivative

Method of u-substitution (Euler)

- Differentiation $\overset{\longleftrightarrow}{\text{inverse}}$ Integration
  - In differentiation : Chain Rule : $\frac{d}{dx}\left[F\big(g(x)\big)\right] = F'\big(g(x)\big)g'(x)$
  - In integration: $\int F'\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + C$

How to apply $\boldsymbol{u}$ −sub



1. Pattern recognition

- ***The integrand is a <u>product of two functions</u>***
  (composite fun * inner fun derivative)
  - f = outer function (usually more complex)
  - $g(x)$ = inner function (usually simpler)

2. Substitute the inner function as u

- $u = g(x)$ then $du = g'(x)dx$
- Include the constant term

3. Rewrite the integral in u: $\int f(u)du$
4. Integrate in u: $\int f(u)du = F(u) + C$
5. Substitute back to x: $\int f\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + C$

$\int x^3\sqrt{2+x^4}\,dx$

$f = \quad \sqrt{2+x^4}$

$g = \quad 2+x^4$

$u = \quad 2+x^4$

$du = \quad 4x^3dx$

$x^3dx = \quad 1/4\,du$

$\int f(u)du =$

$\int \sqrt{\mathbf{2+x^4}}\,\mathbf{x^3dx} = \frac{1}{4}\int \sqrt{u}du$

$F(u) = \frac{1}{4}\left(\frac{2}{3}u^{3/2} + C\right)$

$F(x) = \frac{1}{6}(2+x^4)^{\frac{3}{2}} + C$

Problem

Compute $\int x^3\sqrt{2+x^4}dx$

(a) None of these

(b) $\frac{1}{4}\left(2+x^4\right)^{3/2}+C$

(c) $\frac{1}{6}\left(2+x^4\right)^{3/2}+C$

(d) $\frac{8}{3}\left(2+x^4\right)^{3/2}+C$

(e) $\frac{3}{8}\left(2+x^4\right)^{3/2}+C$

Rewrite : $\int \sqrt{2+x^4}\left(\boldsymbol{x^3dx}\right)$

Id f and g'

- $f =$ $\sqrt{g}$
- $g =$ $2+x^4$
- $g' =$ $4\boldsymbol{x^3}$

u-sub

$u = 2+x^4$

$du = 4\boldsymbol{x^3dx}$

$\boldsymbol{x^3dx} = \frac{1}{4}du$

Solve

$$\int \sqrt{2+x^4}\left(\boldsymbol{x^3dx}\right)$$

$$= \int \sqrt{u}\left(\frac{1}{4}du\right)$$

$$= \frac{1}{4}\int u^{\frac{1}{2}}du$$

$$= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]+C$$

$$= \frac{1}{6}\left(2+x^4\right)^{\frac{3}{2}}+C$$

Solution 2

Differentiate (a) to (e)

Problem

Evaluate $\int x^3\sqrt{x^2+1}\,dx.$

(a) $\frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

(b) $\frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

(c) $3x^2\sqrt{x^2+1} + \frac{x^4}{\sqrt{x^2+1}} + C$

(d) $\frac{2}{5}(x^2+1)^2 - \frac{2}{3}(x^2+1) + C$

(e) None of these

Rewrite : $\int x^2\sqrt{x^2+1}(xdx)$

Id f and g'

- $f =$
- $g =$
- $g' =$

$f = g\sqrt{g+1}$
$g = x^2$
$g' = 2xdx$

u-sub

$u = x^2+1 \Rightarrow x^2 = u-1$
$du = 2xdx$
$\Rightarrow xdx = \frac{1}{2}du$

Evaluate the integral:

$$\int x^2\sqrt{x^2+1}(xdx)$$
$$= \int (u-1)\sqrt{u}\left(\frac{1}{2}du\right)$$
$$= \frac{1}{2}\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$
$$= \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + C$$
$$= \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1) + C$$

Problem

$$\int \frac{\sin x}{(1+\cos x)^3}\,dx =$$

(a) $\dfrac{1}{2(1+\cos x)^2} + C$

(b) $\dfrac{1}{(1+\cos x)^2} + C$

(c) $\dfrac{-1}{2(1+\cos x)^2} + C$

(d) $\dfrac{1}{4(1+\cos x)^4} + C$

(e) $\dfrac{-1}{4(1+\cos x)^4} + C$

Rewrite : $\int \frac{1}{(1+\cos x)^3}(\sin x\, dx)$

Id f and g'

- $f =$
- $g =$
- $g' =$

$f = \frac{1}{(g)^3}$

$g = 1 + \cos x$

$g' = -\sin x\, dx$

u-sub

$u = 1 + \cos x$

$du = -\sin x\, dx$

$\Rightarrow \sin x\, dx = -du$

Evaluate the integral:

$$\int \frac{1}{(1+\cos x)^3}(\sin x\, dx)$$

$$= -\int \frac{1}{u^3}\,du$$

$$= \frac{1}{2u^2} + C$$

$$= \frac{1}{2(1+\cos x)^2} + C$$

$\boldsymbol{u}$ −substitution for a definite integral

Change of Variables for Definite Integrals

❑ **Back substitution (Not recommended)**

If $\int_a^b f\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + C$ then

$$\int_a^b f\big(g(x)\big)g'(x)dx = \big[F\big(g(x)\big)\big]_a^b = F\big(g(a)\big) - F\big(g(b)\big) = [F(u)]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)du$$

❑**Complete substitution** $\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$

- Substitute $u = g(x)$ then $du = g'(x)dx$
- **Change the limits with respect to u**: $\int_a^b \rightarrow \int_{g(a)}^{g(b)}$ (g should be one to one)
- Rewrite the integral in u: $\int_{g(a)}^{g(b)} f(u)\,du$
- Integrate in u: $\int_{g(a)}^{g(b)} f(u)\,du = [F(u)]_{g(a)}^{g(b)} = F\big(g(b)\big) - F\big(g(a)\big)$

Example: Evaluate $\int_1^e \frac{\ln x}{x}dx$ by back substitution.

Find an antiderivative:

Rewrite : $\int \ln x\left(\frac{1}{x}dx\right)$

u-sub $u = \ln x\,;\, du = \frac{1}{x}dx$

Antiderivative $\int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

Evaluate the integral: $\int_1^e \frac{\ln x}{x}dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}(\ln e)^2 - \frac{1}{2}(\ln 1)^2$

$$= \frac{1}{2}(1) - \frac{1}{2}(0) = \frac{1}{2}$$

$\boldsymbol{u}$ −substitution for a definite integral

Evaluate $\int_1^e \frac{\ln x}{x}\,dx$ by complete substitution.

Rewrite : $\int \ln x \left(\frac{1}{x}\,dx\right)$

Id $f(g)$ and g'

- $f(g) =$ $\ln x \Rightarrow f(x) = x$
- $g =$ $\ln x$
- $g' =$ $1/x$

u-sub $u = \ln x\,;\, du = \frac{1}{x}\,dx$

complete substitution for the limits $\int_{x=1}^{x=e} \Rightarrow \int_{u=\ln 1}^{u=\ln e}$

$\int_1^e \ln x \left(\frac{1}{x}\,dx\right) = \int_{\ln 1}^{\ln e} u(du)$

Evaluate the integral:

$= \left[\frac{1}{2}u^2\right]_0^1$

$= \frac{1}{2}[1^2 - 0^2]$

$= \frac{1}{2}$

Problem

Compute $\int_0^{\sqrt{\pi}} x\sin(\pi - x^2)\,dx$

(a) $-\frac{\sin\sqrt{\pi}}{2}$
(b) -2
(c) -1
(d) 1
(e) 2

Rewrite :

$$\int_0^{\sqrt{\pi}} \sin(\pi - x^2)\,(xdx)$$

Id f and g'
- $f =$
- $g =$
- $g' =$

$$f = \sin x$$
$$g = \pi - x^2$$
$$g' = -2x$$

u-sub

$$u = \pi - x^2$$
$$du = -2xdx$$
$$\Rightarrow xdx = -\frac{1}{2}du$$

complete substitution for the limits

$$\int_{x=0}^{x=\sqrt{\pi}} \Rightarrow \int_{\pi-0^2}^{\pi-\sqrt{\pi}^2}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_\pi^0 \sin u\left(-\frac{1}{2}du\right)$$

Evaluate the integral:

$$= \frac{1}{2}\int_0^\pi \sin u\,du$$
$$= \frac{1}{2}[-\cos u]_0^\pi$$
$$= \frac{1}{2}[-\cos\pi + \cos 0] = 1$$

Problem

Evaluate $\displaystyle\int_0^{\pi/4} \frac{\sec^2(\theta)}{2+\tan(\theta)}\,d\theta$

(a) $\ln\left(\frac{4}{3}\right)$

(b) $\ln\left(\frac{\pi}{4}\right)$

(c) $\ln\left(\frac{\pi}{8}\right)$

(d) $\ln\left(\frac{\pi}{12}\right)$

(e) $\ln\left(\frac{3}{2}\right)$

Rewrite : $\int_0^{\pi/4} \frac{1}{2+\tan\theta}\left(\sec^2\theta\, d\theta\right)$

Id f and g'

- $f =$
- $g =$
- $g' =$

$f = 1/g$

$g = 2 + \tan\theta$

$g' = \sec^2\theta$

u-sub

$u = 2 + \tan\theta$

$du = \sec^2\theta\, d\theta$

complete substitution for the limits $\int_{x=0}^{x=\pi/4} \Rightarrow \int_{2+\tan 0}^{2+\tan \pi/4}$

Evaluate the integral:

$\int_2^3 \frac{1}{u}(du)$

$= [\ln|u|]_2^3$

$= \ln 3 - \ln 2$

$= \ln\frac{3}{2}$

Problem

If f is continuous and $\int_0^{16} f(x)\,dx = 8$,

find $\int_0^4 x f(x^2)\,dx$.

(a) 16

(b) 2

(c) 8

(d) 64

(e) 4

Rewrite :

$\int_0^4 f(x^2)\,(xdx)$

Id f and g'

-
- $g =$
- $g' =$

$g = x^2$

$g' = 2x$

u-sub

$u = x^2$

$du = 2xdx$

$\Rightarrow xdx = \frac{1}{2}du$

complete substitution for the limits

$\int_{x=0}^{x=4} \Rightarrow \int_{u=0^2}^{u=4^2}$

$\int_0^{16} f(u)\left(\frac{1}{2}du\right)$

Evaluate the integral:

$= \frac{1}{2}\int_0^{16} f(u)du$

$= \frac{1}{2}(8)$

$= 4$

Problem

Evaluate $\int_{\sqrt{2}}^{2} \frac{4x}{x^2-1}\,dx$

(a) $2\ln 3$

(b) $2\ln 2 - 2\ln\sqrt{2}$

(c) $\ln 3 - 1$

(d) $2\ln 3 - 2$

(e) $2\ln 2$

Rewrite :

$2\int_{\sqrt{2}}^{2} f(x^2)\,(2xdx)$

Id f and g'

-
- $g =$
- $g' =$

$g = x^2 - 1$
$g' = 2x$

u-sub

$u = x^2 - 1$
$du = 2xdx$

complete substitution for the limits

$\int_{x=\sqrt{2}}^{x=2} \Rightarrow \int_{u=1}^{u=3}$

Evaluate the integral:

$2\int_{1}^{3} \frac{1}{u}\,du$
$= 2[\ln|u|]_1^3$
$= 2\ln 3$

Problem

Evaluate $\displaystyle\int_{\pi^2/16}^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

(a) $\frac{\sqrt{2}}{2}+1$

(b) $2+\sqrt{2}$

(c) $\frac{\sqrt{2}}{2}-1$

(d) $-2+\sqrt{2}$

(e) $1-\sqrt{2}$

Rewrite : $\int_{\pi^2/16}^{\pi^2} f(\sqrt{x})\left(\frac{1}{\sqrt{x}}\,dx\right)$

Id f and g'

-
- $g =$
- $g' =$

$g = \sqrt{x}$

$g' = \frac{1}{2\sqrt{x}}$

u-sub

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2du$

complete substitution for the limits

$\int_{x=\pi^2/16}^{x=\pi^2} \Rightarrow \int_{u=\pi/4}^{u=\pi}$

Evaluate the integral:

$\int_{\pi/4}^{\pi} \sin u\,(2du)$

$= 2[-\cos u]_{\pi/4}^{\pi}$

$= 2\left(1+\frac{\sqrt{2}}{2}\right)$

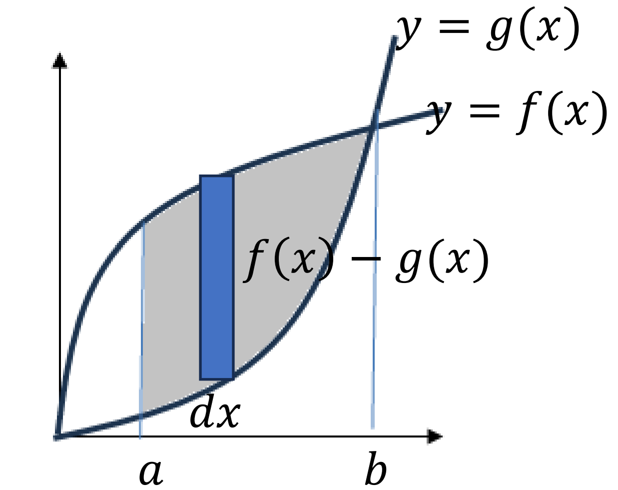
Area between curves

The integration domain is x axis $\Rightarrow dx$

1. Plot the graphs
2. Draw an infinitesimal strip

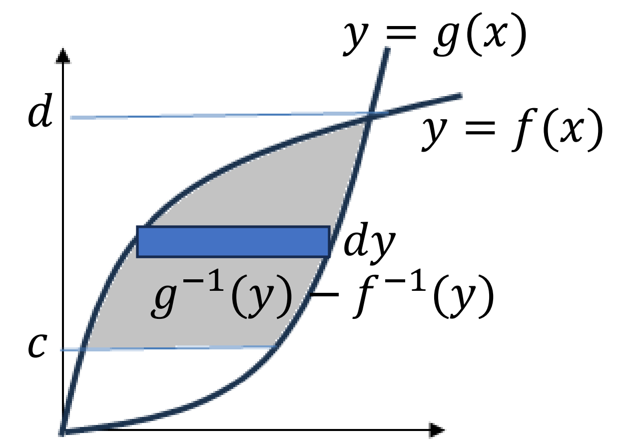
3. Find the area of the infinitesimal strip
   - [Function of x]dx
   - [Large fun – Small fun]dx
   - $[f(x) - g(x)]dx$

4. Find the upper/lower limits
   - Solve the equations if needed
   - In this case, $[a, b]$ is given
5. Integrate the area of infinitesimal strips
   - $\int_a^b [f(x) - g(x)]dx$



The integration domain is y axis $\Rightarrow dy$

2. the area of the infinitesimal strip
   - [Function of y]dy
   - [Large fun, g – Small fun,f]dy
   - $[g^{-1}(y) - f^{-1}(y)]dy$
   - $\int_c^d [g^{-1}(y) - f^{-1}(y)]dy$

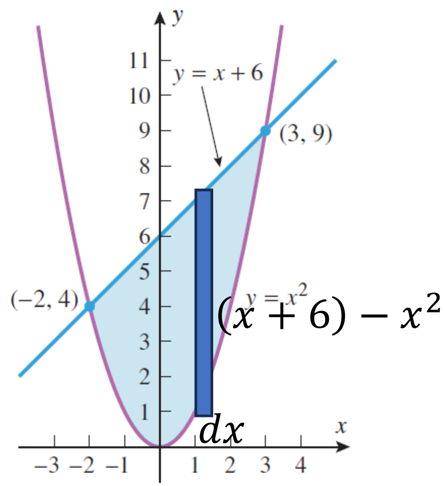


Area between curves $\boldsymbol{f(x)}$ and $\boldsymbol{g(x)}$, $\boldsymbol{f > g}$ (Example)

Find the area between the curves: $y = x^2$ and $y = x + 6.$

When $f(x) \geq g(x)$

1. Plot the graphs



2. Draw an infinitesimal strip

3. Find the area of the infinitesimal strip

4. Find the upper/lower limits

5. Integrate the area of infinitesimal strips

$[(x+6) - x^2]dx$

(Need limits for x)

$x^2 = x + 6$

$(x+2)(x-3) = 0$

$\Rightarrow x = -2{,}3$

$= \int_{-2}^{3} (x + 6 - x^2)dx$

$= \left[\frac{1}{2}x^2 - 6x - \frac{1}{3}x^3\right]_{-2}^{3}$

$= \frac{1}{2}[x^2]_{-2}^{3} - 6[x]_{-2}^{3} - \frac{1}{3}[x^3]_{-2}^{3}$

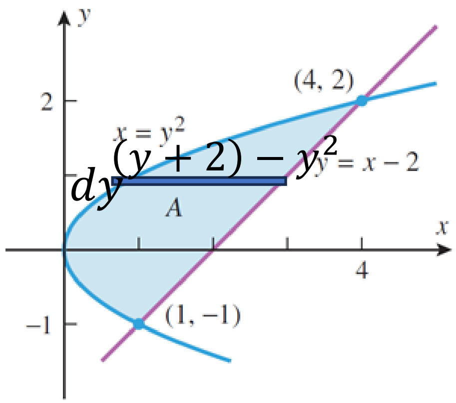
$= \frac{9-4}{2} - 6(3-(-2)) - \frac{27-(-8)}{3}$

$= -\frac{235}{6}$

Area between curves $f(y)$ and $g(y)$, $f > g$ (Example)

When $f(y) \geq g(y)$

1. Plot the graphs



Area between
$x = y^2$ and $y = x - 2$

2. Draw an infinitesimal strip

3. Find the area of the infinitesimal strip

$[y + 2 - y^2]dy$

(Need limits for y)

4. Find the upper/lower limits

- $y + 2 = y^2;\ y^2 - y - 2 = 0$

$$(y+1)(y-2) = 0$$

$$\Rightarrow y = -1,2$$

5. Integrate the area of infinitesimal strips

$\int_{-1}^{2}(y + 2 - y^2)\,dy$

$= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$

$= \frac{1}{2}(2^2 - 1) + 2(2+1) - \frac{1}{3}(2^3 + 1)$

$= \frac{3}{2} + 6 - \frac{9}{3} = \frac{3}{2} + 3 = \frac{9}{2}$

Problem

Find the area bounded by
$y = e^x$, $y = e^{-x}$, $x = 0$, and $x = 1$.

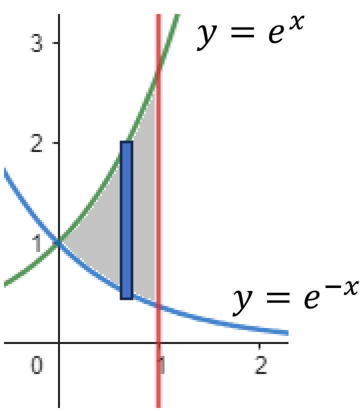
(a) $e + \frac{1}{e} - 2$

(b) $e - \frac{1}{e}$

(c) $e + \frac{1}{e} + 2$

(d) $1 + \frac{1}{e}$

(e) $1 + \frac{1}{e} - 2$

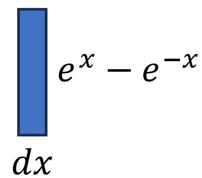


When $f(y) \geq g(y)$

1. Plot the graphs

2. Draw an infinitesimal strip

3. Find the area of the infinitesimal strip

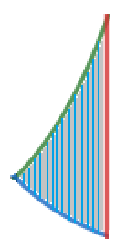


$$A(|) = (e^x - e^{-x})dx$$

4. Find the upper/lower limits

$$0 \leq x \leq 1$$

5. Integrate the area of infinitesimal strips



$$\int_0^1 (e^x - e^{-x})dx$$
$$= [e^x + e^{-x}]_0^1$$
$$= (e^1 + e^{-1}) - (1 + 1)$$
$$= e + \frac{1}{e} - 2$$

Problem

Which of the following integrals gives the area of the region bounded by the curves $x = y^2$ and $x = 6 - y$?

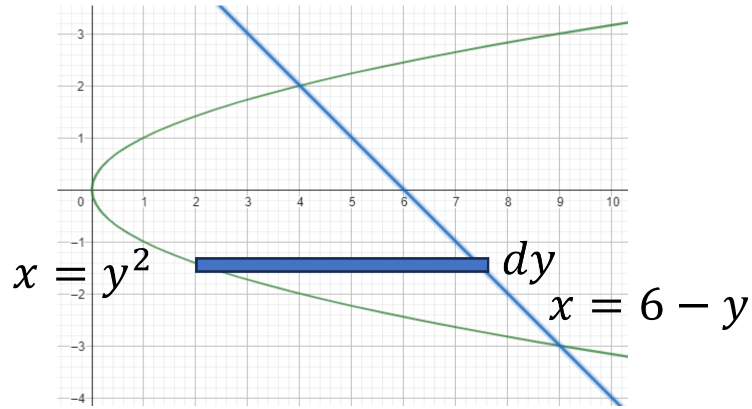
(a) $\int_{-3}^{2} (6 - y - y^2)\, dy$

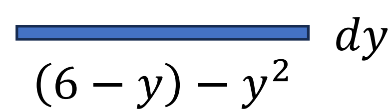
(b) $\int_{-3}^{2} (y^2 - 6 + y)\, dy$

(c) $\int_{4}^{9} (6 - x - \sqrt{x})\, dy$

(d) $\int_{4}^{9} (\sqrt{x} - 6 + x)\, dy$

(e) $\int_{4}^{9} (6 - y - y^2)\, dy$





Integrand = $(6 - y - y^2)dy$

Limits = y intersections

$6 - y - y^2 = 0$

$y^2 + y - 6 = 0$

$(y + 3)(y - 2) = 0$

$y = -3,2 \Rightarrow \int_{-3}^{2}$

(a) $\int_{-3}^{2}(6 - y - y^2)dy$

Problem

Find the area bounded by $y + x^2 = 6$ and $y + 2x - 3 = 0$.

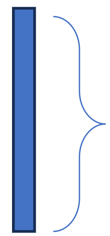
(a) $\frac{16}{3}$

(b) $\frac{40}{3}$

(c) $\frac{20}{3}$

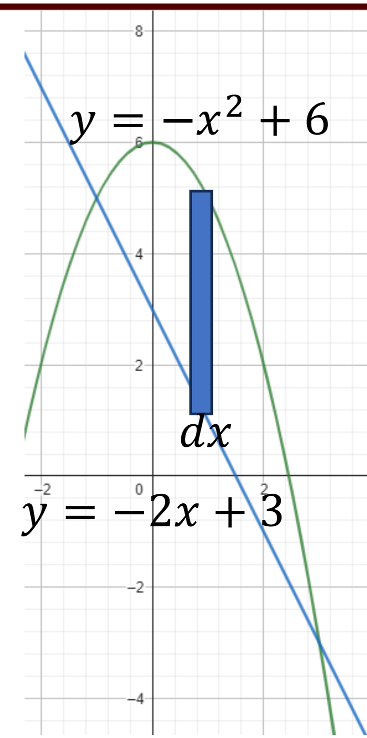
(d) $\frac{32}{3}$

(e) None of the above



$(-x^2 + 6) - (-2x + 3)$

dx



Integrand = $(-x^2 + 2x + 3)dy$

Limits = y intersections

$-x^2 + 2x + 3 = 0$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = -3, -1 \Rightarrow \int_{-1}^{3}$

$\int_{-1}^{3}(-x^2 + 2x + 3)dy = \frac{1}{6}\left(3 - (-1)\right)^3 = \frac{64}{6} = \frac{32}{3}$

Formula:

$$-\int_a^b (x - a)(x - b)dx = \frac{(b-a)^3}{6}$$

Problem

Find the area of the region bounded by $x = y^2$ and $x = y + 2$.

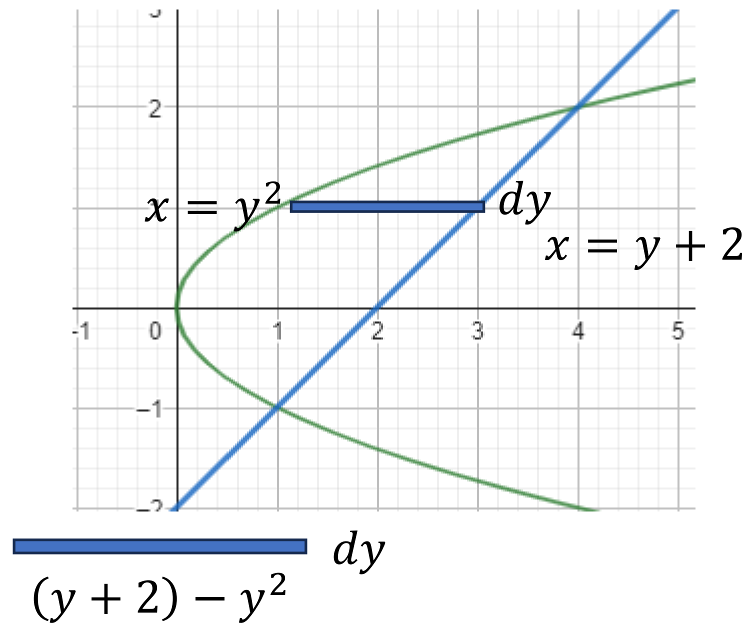
(a) $\frac{9}{2}$

(b) $\frac{3}{2}$

(c) $\frac{19}{6}$

(d) $\frac{16}{3}$

(e) None of the above



Integrand = $(2 + y - y^2)dy$

Limits = y intersections

$y^2 - y - 2 = 0$

$(y+1)(y-2) = 0$

$y = -1,2 \Rightarrow \int_{-1}^{2}$

$-\int_{-1}^{2}(y+1)(y-2)dy = \frac{1}{6}(2+1)^3 = \frac{27}{6} = \frac{9}{2}$