Math 251 Fall 2024 Chapter 16

Use this as a guide to help you organize your thoughts on Chapter 16. You also should review section 14.7, relative maximum and minimum of a function $f(x, y)$.

As you begin to prepare for your final exam, hand write a few examples done in lecture over each concept numbered below. Next, completely randomize every problem done in lecture from chapter 16. Once you feel prepared, take the 'Former Final Exams' I posted in the top module in Canvas. Work it in a *timed* 2 hour time, and at the end of the 2 hours, use the posted solutions to see what you need to work on.

NO FORMULA SHEET CAN BE USED ON THE FINAL EXAM!

1. If f is defined on a smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \leq t \leq b$, then the line integral of f along C is
$$\int_C f(x,y)ds = \int_a^b (f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b (f(x(t), y(t))|\mathbf{r}'(t)|\,dt.$$

2. If f is defined on a smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$, then the line integral of f along C is
$$\int_C f(x,y,z)ds = \int_a^b (f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b (f(x(t), y(t), z(t))|\mathbf{r}'(t)|\,dt.$$

3. Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.

   a.) The line integral of f along C with respect to x is $\int_C f(x,y)dx = \int_a^b (f(x(t), y(t))\ x'(t)\ dt.$

   b.) The line integral of f along C with respect to y is $\int_C f(x,y)dy = \int_a^b (f(x(t), y(t))\ y'(t)\ dt.$

4. Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \leq t \leq b$.

   The line integral of f along C with respect to z is $\int_C f(x,y,z)dz = \int_a^b (f(x(t), y(t), z(t))\ z'(t)\ dt.$

5. The line integral of $\mathbf{F}$ along a curve C parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$, is $\int_c \mathbf{F}\cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt.$

6. Test for conservative vector fields

   a.) $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = P\mathbf{i}+Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

   b.) If $\mathbf{F}$ is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

7. Fundamental Theorem for Line Integrals: Let C be a smooth curve parameterized by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C.

   Then $\displaystyle\int_c \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

8. Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then $\displaystyle\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$.

9. If $\mathbf{F} = \langle P, Q, R\rangle$ is a a vector field on $\Re^3$ and the partial derivatives of P, Q, and R all exist, the **del operator**, denoted by ∇, is $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle$.

10. The Divergence of F is $\nabla\cdot\mathbf{F} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right\rangle \cdot \langle P, Q, R\rangle$.

11. The Curl of $\mathbf{F}$ is the vector field on $\Re^3$ defined by curl $\mathbf{F} = \nabla \times \mathbf{F}$.

12. If a smooth parametric surface S is parameterized by $\mathbf{r}(u,v)$, and S is covered just once as (u,v) ranges throughout the parametric domain D, then the **surface area** of S is $\displaystyle A(S) = \iint\limits_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$.

13. If a smooth parametric surface S is parameterized by $\mathbf{r}(u,v)$, and S is covered just once as (u,v) ranges throughout the parametric domain D, then the surface integral of f over S is given by $\displaystyle\iint_S f(x,y,z)\, dS = \iint\limits_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$.

14. Let $\mathbf{F}$ be a vector field whose domain includes the positively oriented surface S, where S is defined parametrically by $\mathbf{r}(u,v)$, $u, v \in D$. Then the surface integral of $\mathbf{F}$ over S, also called the **Flux** of $\mathbf{F}$ over S, is $\displaystyle\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA$.

15. **Stokes' Theorem:** Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive (counterclockwise) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains S. Then $\displaystyle\int\limits_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$

16. **Divergence Theorem** Let E be a simple solid region whose boundary surface has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then $\displaystyle\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_E \text{div } \mathbf{F}\, dV$