04

MATH 152
Week in Review

EXAM I Review

(5.5 through 7.2)

Compute $\int_0^{\sqrt{\pi}} x\sin(\pi - x^2)\,dx$

(a) $-\frac{\sin\sqrt{\pi}}{2}$

(b) -2

(c) -1

(d) 1 ← correct

(e) 2

Rewrite : $\int_0^{\sqrt{\pi}} \sin(\pi - x^2)\,(xdx)$

Id f and g'

- $f =$
- $g =$
- $g' =$

$f = \sin x$

$g = \pi - x^2$

$g' = -2x$

u-sub

$u = \pi - x^2$

$du = -2xdx$

$\Rightarrow xdx = -\frac{1}{2}du$

complete substitution for the limits $\int_{x=0}^{x=\sqrt{\pi}} \Rightarrow \int_{\pi-0^2}^{\pi-\sqrt{\pi}^2}$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

Evaluate the integral:

$\int_\pi^0 \sin u\left(-\frac{1}{2}du\right)$

$= \frac{1}{2}\int_0^\pi \sin u\,du$

$= \frac{1}{2}[-\cos u]_0^\pi$

$= \frac{1}{2}[-\cos\pi + \cos 0] = 1$

Compute $\int_1^2 x\ln(x^2)\,dx$.

(a) $\frac{\ln 4}{2}$

(b) $\ln 4$

(c) $4\ln 4 - 3$

(d) $\frac{3}{2}$

(e) $\ln 16 - \frac{3}{2}$ $\leftarrow$ correct

$$u\text{-sub}: u = x^2 \quad \Rightarrow du = 2xdx$$

$$= \frac{1}{4}\int \ln u\, du$$

$$= \frac{1}{4}[u \ln|u| - u] + C$$

$$= \frac{1}{4}[x^2 \ln x^2 - x^2] + C$$

Which of the following integrals gives the area of the region bounded by the curves $x = y^2$ and $x = 6 - y$?

(a) $\int_{-3}^{2} (6 - y - y^2)\,dy$ $\leftarrow$ correct

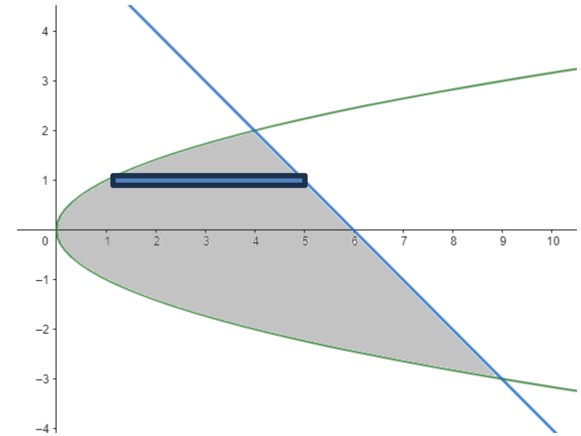
(b) $\int_{-3}^{2} (y^2 - 6 + y)\,dy$

(c) $\int_{4}^{9} (6 - x - \sqrt{x})\,dy$

(d) $\int_{4}^{9} (\sqrt{x} - 6 + x)\,dy$

(e) $\int_{4}^{9} (6 - y - y^2)\,dy$

Plot



Slice

$A(\text{===}) = [(6 - y) - (y^2)]dy$

Intersections

$6 - y = y^2$

$y^2 + y - 6 = 0$

$(y - 2)(y + 3) = 0$

$y = -3, 2$

Area between curve

$\int_{-3}^{2} (6 - y - y^2)dy$

The region bounded by $y = e^x$ and the x-axis on the interval $[0, 2]$ is rotated about the x-axis. Find the volume of the resulting solid.

(a) $\frac{\pi e^4}{2}$

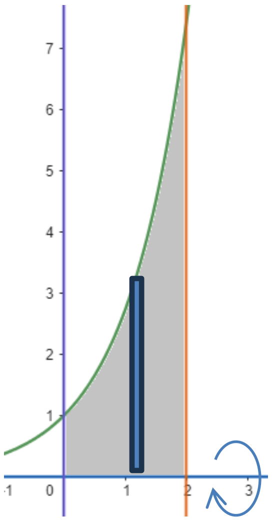
(b) $\frac{\pi e^2}{2}$

(c) $\frac{\pi}{2}(e^4 - 1)$ ← correct

(d) $\frac{\pi}{2}(e^2 - 1)$

(e) $2\pi(e^4 - 1)$

Plot



Slice

$V($  $) = \pi(e^x)^2 dx$

$\pi e^{2x} dx$

limit

$dx \in [0,2]$

Volume

$\int_0^2 \pi e^{2x} dx$

$= \pi \left[\frac{1}{2} e^{2x}\right]_0^2$

$= \frac{\pi}{2}(e^4) - 1$

Consider the region bounded by the curves $x = y^2 - 2y$ and the y-axis. Which of the following represents the volume of solid formed when the region is rotated about $y = 4$?

(a) $\int_0^2 2\pi y(y^2 - 2y)\, dy$

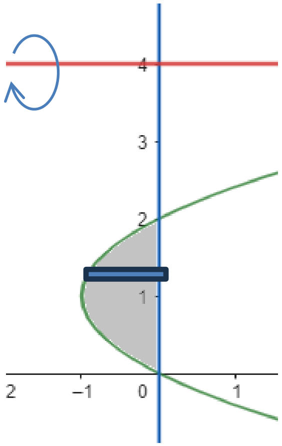
(b) $\int_0^2 2\pi y(2y - y^2)\, dy$

(c) $\int_0^2 2\pi(4 - y)(y^2 - 2y)\, dy$

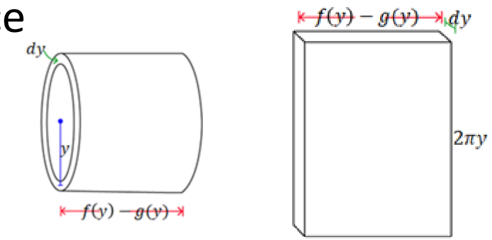
(d) $\int_0^2 \pi(y - 4)(4y^2 - y^4)\, dy$

(e) $\int_0^2 2\pi(4 - y)(2y - y^2)\, dy$ ← correct

Plot



Slice



$V(\ ) = 2\pi(4 - y)(0 - [y^2 - 2y])dy$

limit

$dy \in [0,2]$

Volume

$\int_0^2 2\pi(4 - y)(2y - y^2)dy$

Consider the region bounded by the two curves $y = \cos x$, $y = \sin x$ and the two lines $x = 0$ and $x = \frac{\pi}{4}$.
Which of the following represents the volume of this region being rotated about the line $x = -1$?

(a) $\int_0^{\frac{\pi}{4}} 2\pi(x+1)(\cos x - \sin x)\,dx$ ← correct

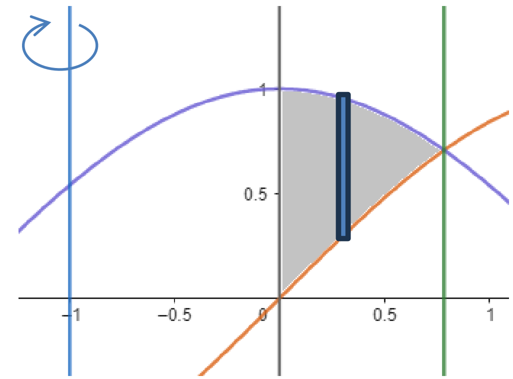
(b) $\int_0^{\frac{\pi}{4}} 2\pi(x+1)(\sin x - \cos x)\,dx$

(c) $\int_{-1}^{\frac{\pi}{4}} 2\pi(x+1)(\cos x - \sin x)\,dx$

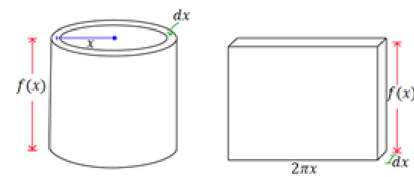
(d) $\int_0^{\frac{\pi}{4}} 2\pi(x+1)(\cos^2 x - \sin^2 x)\,dx$

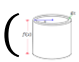
(e) $\int_0^{\frac{\pi}{4}} \pi(\cos^2 x - \sin^2 x)\,dx$

Plot



Slice



$V($  $) = 2\pi(x-(-1))(\cos x - \sin x)dx$

limit

$dx \in \left[0, \frac{\pi}{4}\right]$

Volume

$\int_0^{\pi/4} 2\pi(x+1)(\cos x - \sin x)dx$

Find the area of the region determined by the curve $f(x) = x \sin x$ and the x-axis on the interval $[0, \pi]$.

(a) 1

(b) π ← correct

(c) $\frac{\pi}{2}$

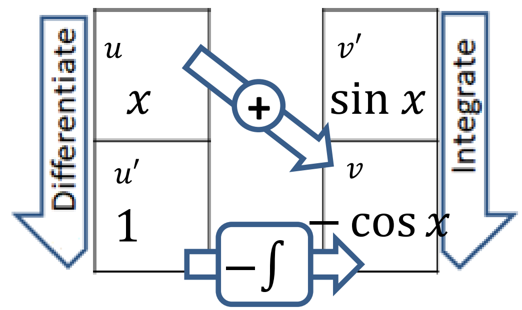
(d) $\pi - 1$

(e) $-\pi$

$$\int_0^{\pi} |x \sin x| \, dx$$
$$= \int_0^{\pi} x \sin x \, dx$$

u----**L I A T E**----v'

$x \quad \sin x$



$$= -[x \cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx$$
$$= -[\pi \cos \pi - 0]_0^{\pi} + [\sin \pi]_0^{\pi}$$
$$= \pi$$

Which of the following integrals gives the volume of the solid obtained by rotating the region bounded by $y = 5 - x^2$ and $y = 1$ about the x-axis.

(a) $\pi \int_{-2}^{2} \left(1 - (5 - x^2)^2\right) dx$

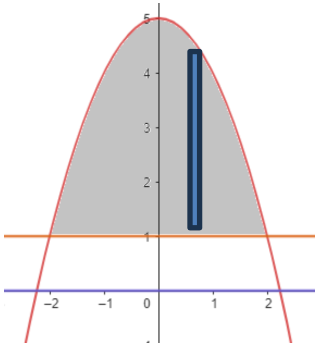
(b) $\pi \int_{-2}^{2} (4 - x^2)^2 \, dx$

(c) $2\pi \int_{-2}^{2} x(4 - x^2) \, dx$

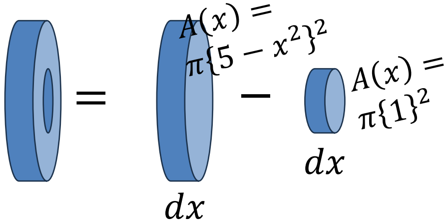
(d) $\pi \int_{-2}^{2} \left((5 - x^2)^2 - 1\right) dx$ ← correct

(e) $2\pi \int_{-2}^{2} x(x^2 - 4) \, dx$

Plot



Slice



$V($  $) = \pi(5 - x^2)^2 dx - \pi(1)^2 dx$

$\pi[(5 - x^2)^2 - 1]dx$

Limit

$5 - x^2 = 1 \quad \Rightarrow x^2 = 4 \quad \Rightarrow x = \pm 2$

$dx \in [-2,2]$

Volume

$\int_{-2}^{2} \pi[(5 - x^2)^2 - 1]dx$

Find the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $x = y^3$ around the y-axis.

(a) $\frac{\pi}{35}$

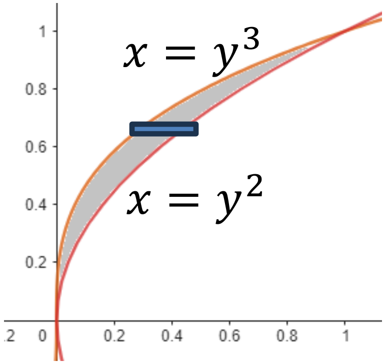
(b) $\frac{\pi}{10}$

(c) $\frac{\pi}{12}$

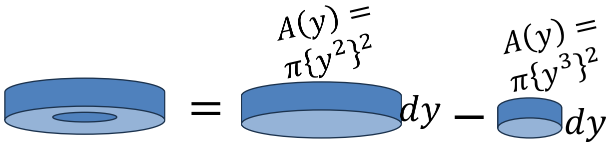
(d) $\frac{2\pi}{35}$ ← correct

(e) $\frac{\pi}{105}$

Plot



Slice



$V($$) = \pi(y^2)^2 dy - \pi(y^3)^2 dy$

$\pi[y^4 - y^6]dy$

Limit

$y^2 = y^3 \quad \Rightarrow x = 0,1$

$dy \in [0,1]$

Volume

$\int_0^1 \pi(y^4 - y^6)dy$

$= \pi\left[\frac{1}{5}y^5 - \frac{1}{7}y^7\right]_0^1 = \pi\left[\frac{1}{5} - \frac{1}{7}\right] = \frac{2\pi}{35}$

An ideal spring has a natural length of 10 meters. The work done in stretching the spring from 14 meters to 18 meters is 24J. Determine the spring constant k.

(a) $k = \frac{1}{2}$ N/m
(b) $k = \frac{3}{8}$ N/m
(c) $k = 1$ N/m ← correct
(d) $k = 3$ N/m
(e) $k = 6$ N/m

$F(x) = kx$

$dW = F(x)dx = kxdx$

Work done from x_0 to x_1 (from resting length)

$W = \int_{x_0}^{x_1} kxdx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_0^2$

Work done from 14 to 18 (spring length)

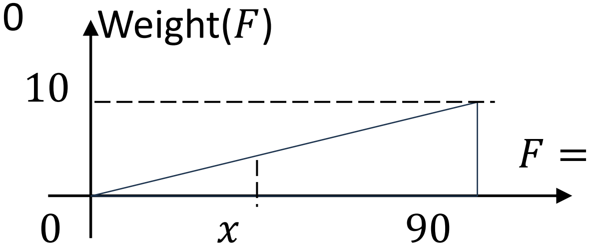
Work done from 4 to 8 (from resting length)

$$24 = \frac{1}{2}k8^2 - \frac{1}{2}k4^2$$
$$= \frac{k}{2}(8-4)(8+4)$$
$$= \frac{k}{2}4 \cdot 12 = k24$$
$$k = 1$$

A 90 ft cable weighing 10 lb is hanging down the side of a 200 ft building. How much work is required to pull the rope 30 feet up the side of the building?

(a) 6000 ft-lb
(b) 1500 ft-lb
(c) 250 ft-lb ← correct
(d) 300 ft-lb
(e) 50 ft-lb

- **Step 1** : plot a graph in the coordinate system (**weight vs length**)
  - Set the top of the rope = 0



$F = \left(\frac{10}{90}\right)x$

- **Step 2:** Slicing the cable by dx segment and consider a segment at location x (to be lifted by x)
  - Find the weight of rope with length x (=force, F)
    - $F(x) = \frac{10}{90}x$
- **Step 4.** Find the work done by lifting a cable at x by the length of dxfts.
  - $dW = F(x)dx = \left[\frac{10}{90}dx\right]x = \frac{1}{9}xdx$
- **Step 5.** Find the total work by integrating dW
  - $W = \int_{60}^{90} \frac{1}{9}xdx = \frac{1}{2}\cdot\frac{1}{9}[x^2]_{60}^{90}$

$$= \frac{1}{2}\cdot\frac{1}{9}[90^2 - 60^2] = \frac{(90-60)(90+60)}{2\cdot 9} = \frac{9(30-20)(30+20)}{2\cdot 9} = \frac{9\cdot 500}{2\cdot 9} = 250$$

The solid S has a triangular base with vertices $(-1,0)$, $(1,0)$, and $(0,2)$. Cross sections perpendicular to the x-axis are squares. Find the volume of S.

(a) $\frac{4}{3}$

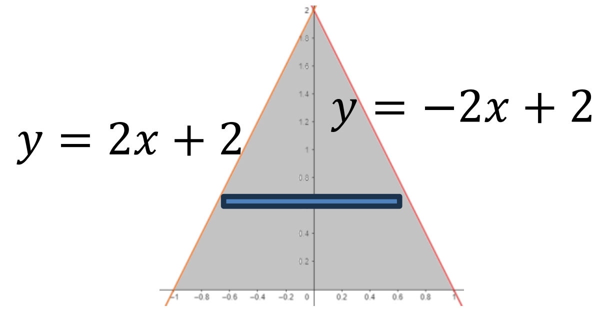
(b) $\frac{8}{3}$ ← correct

(c) $\frac{1}{3}$

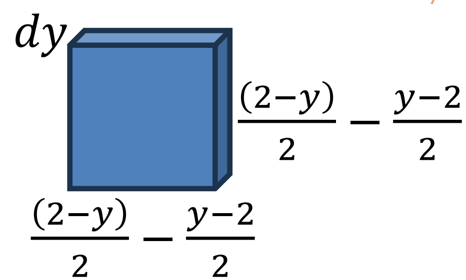
(d) $\frac{2}{3}$

(e) $\frac{5}{3}$

Plot



Slice



$V($$) = (y-2)^2 dy$

limit

$dy \in [0,2]$

Volume

$$\int_0^2 (y-2)^2 dy$$

$$= \frac{1}{3}\left[(y-2)^3\right]_0^2$$

$$= \frac{1}{3}\left[0-(-2)^3\right]$$

$$= \frac{8}{3}$$

Compute $\int_0^1 \arctan x \, dx$.

(a) $\frac{\pi}{4} - \frac{1}{2}\ln 2$ ← correct

(b) $\frac{\pi}{4} - \ln 2$

(c) $1 - \frac{1}{2}\ln 2$

(d) $1 - \ln 2$

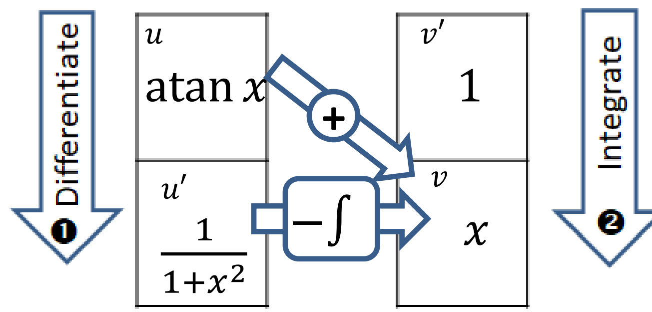
(e) $\frac{\pi}{4}$

Evaluate $\int \tan^{-1} x \, dx$ by the tabular method

Hint: $\int \tan^{-1} x \, dx = \int 1 \cdot \tan^{-1} x \, dx$

u----L I A T E----v'

$\tan^{-1} x \quad 1$



$$\int_0^1 \tan^{-1} x = [x \tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

$$= [x \tan^{-1} x]_0^1 - \frac{1}{2}[\ln(1+x^2)]_0^1 \text{ (u-sub)}$$

$$= (\tan^{-1}(1) - 0) - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Evaluate $\displaystyle\int_0^1 \frac{x^2}{e^x}\,dx$.

(a) $2 - \frac{5}{e}$ ← correct

(b) $\frac{5}{e} - 2$

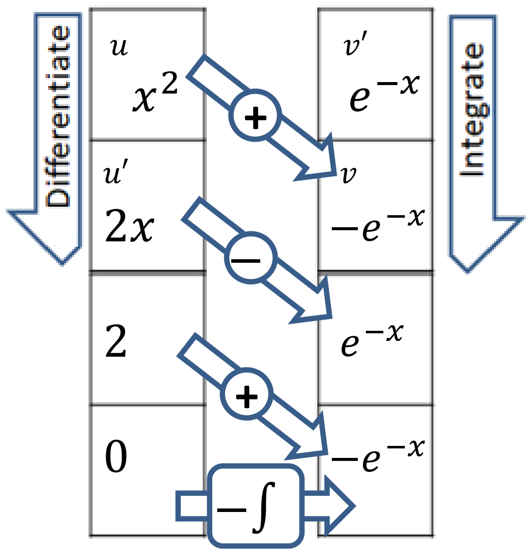
(c) $1 - \frac{3}{e}$

(d) $1 - \frac{2}{e}$

(e) $1 - \frac{1}{e}$

u----L I **A** T **E**----v'

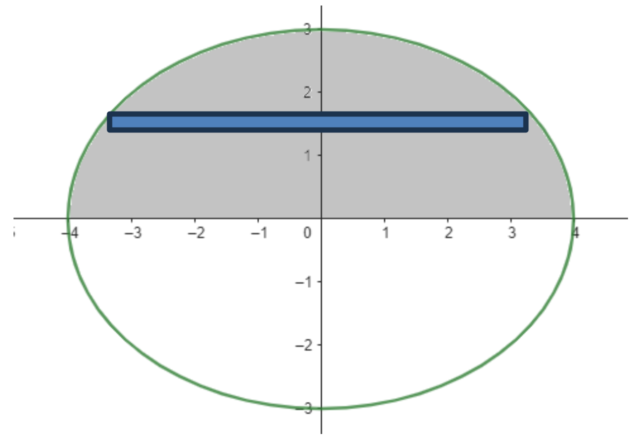
x^2 $\quad e^{-x}$



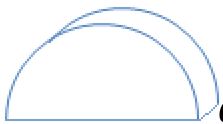
$$\int_0^1 \frac{x^2}{e^2}\,dx = [-x^2e^{-x} - 2xe^{-x} - 2e^{-x}]_0^1$$
$$= (-e^{-1} - 2e^{-1} - 2e^{-1}) - (-2)$$
$$= 2 - 5e^{-1}$$

(10 points) Consider the solid whose base is the upper half of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. Cross sections perpendicular to the y axis are semicircles. Find the volume of the solid.

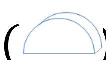
Plot



Slice



$$A(y)=\frac{\pi}{2}[x(y)]^2$$

$V($$)$ $=\frac{\pi}{2}\left[16\left(1-\frac{y^2}{9}\right)\right]dy=8\pi\left(1-\frac{y^2}{9}\right)dy$

limit

$dy\in[0,3]$

Volume

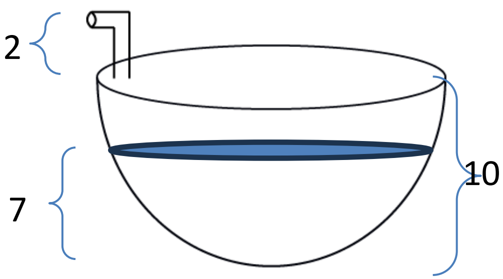
$$\int_0^3 8\pi\left(1-\frac{y^2}{9}\right)dy$$

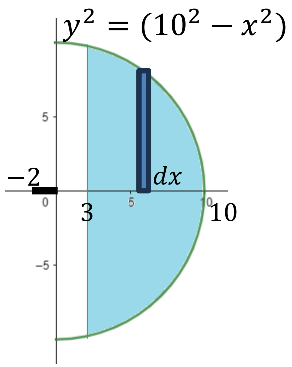
$$=8\pi\left[y-\frac{1}{27}y^3\right]_0^3$$

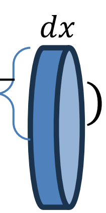
$$=8\pi[3-1]$$

$$=16\pi$$

(10 points) A hemispherical tank has the shape shown below. The tank has a radius of 10 meters with a 2 meter spout at the top of the tank. The tank is filled with water to a depth of 7 meters. The weight density of water is $\rho g = 9800\text{N}/\text{m}^3$. Suppose we want to find the work required to pump the water through the spout







$V(\sqrt{10^2 - x^2}) = \pi r^2 dx$

$= \pi(10^2 - x^2)dx$

The tank shown is full of water. Find the work required to pump the water out of the spout. (Use 9800 N/m^3 as water density)

- **Step 1** : plot a graph in the coordinate system (tank shape vs depth): Set the top of the tank = 0
- **Step 2:** Slicing the tank by dx height (Set the top = 0) and consider a slab at location x (to be lifted by x)
  - Find the volume of the disc at x
    - $dv = \pi(10^2 - x^2)dx$
- **Step 3:** Find the weight of water within the disc (=force, F)
  - **water weight = (water volume)x(weight density)**
    - $dF = \rho dv = 9800\pi(10^2 - x^2)dx$
- **Step 4.** Find the work done by pumping the water disc dF lb by a length of $x + 2$fts (due to spout).
  - $dW = (dF)x = [9800\pi(10^2 - x^2)dx](x+2)$
    $= 9800\pi(10^2 - x^2)(x+2)dx$
- **Step 5.** Find the total work by integrating dW (Limit ??)
  - $W = 9800\pi \int_3^{10} (10^2 - x^2)(x+2)dx$

(7 points) Compute $\int x^5 e^{x^3}\, dx$

Rewrite

$\int x^5 e^{x^3} dx = \int x^3 e^{x^3} (x^2 dx)$

u −sub

$u = x^3 \;\Rightarrow du = 3x^2 dx \quad \Rightarrow x^2 dx = \frac{1}{3} du$

$\int x^3 e^{x^3}(x^2 dx) = \int u e^u \left(\frac{1}{3} du\right)$

$= \frac{1}{3} \int u e^u du$

$= \frac{1}{3} [u e^u - e^u] + C$

$= \frac{1}{3} x^3 e^{x^3} - e^{x^3} + C$

Evaluate $\int x e^x dx$

u----**L**ogarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential----v'

x $\qquad$ e^x

u'-- 1 $\qquad$ e^x--v

$\int x e^x dx = \quad x e^x \quad - \int 1 \cdot e^x dx$

$\int u v' dx = \quad uv \quad - \int u' v dx$

$= x e^x - e^x + C$

(13 points) Consider the region S bounded by the curve $f(x) = e^x$, y-axis, and its tangent line at $x = 1$.

(a) (2 points) Find the tangent line to the curve $f(x) = e^x$ at $(1, e)$.

(b) (3 points) Find the precise area of the region S.

(c) (4 points) The volume of the solid obtained by rotating S about $x = 2$. **Do not evaluate!**

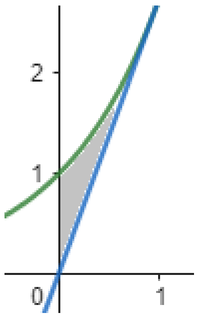
(d) (4 points) The volume of the solid obtained by rotating S about $y = 5$. **Do not evaluate!**

(a) $m = f'(1) = e$

Pt-slope Eq: $y - e = e(x - 1)$

$y = ex$

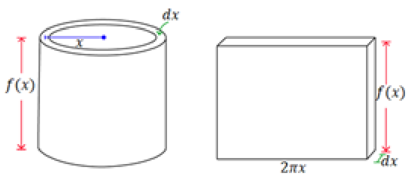
(b) Plot

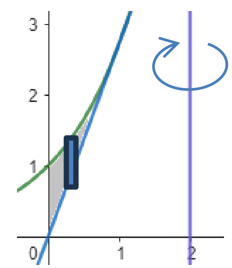


$$\int_0^1 (e^x - ex)dx = \left[e^x - \frac{e}{2}x^2\right]_0^1 = \left(e - \frac{e}{2}\right) - (1 - 0) = \frac{e}{2} - 1$$

(c) Plot

Slice





$V($  $) = 2\pi(2 - x)(e^x - ex)dx$

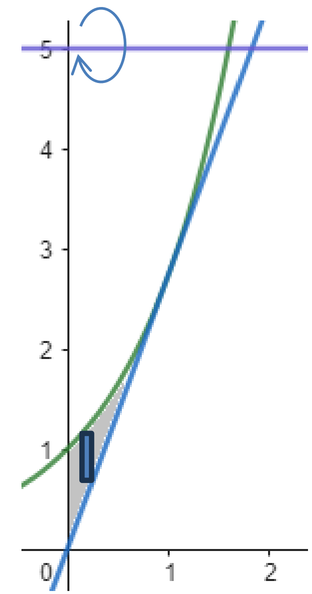
limit

$dx \in [0,1]$

Volume

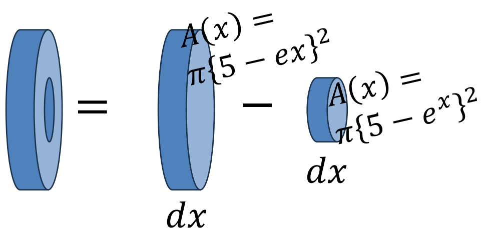
$$\int_0^1 2\pi(2 - x)(e^x - ex)dx$$

(d) (4 points) The volume of the solid obtained by rotating S about $y = 5$. **Do not evaluate!**



Plot

Slice



$V($  $) = \pi(5-ex)^2 dx - \pi(5-e^x)^2 dx$

$\pi[(5-ex)^2 - (5-e^x)^2]dx$

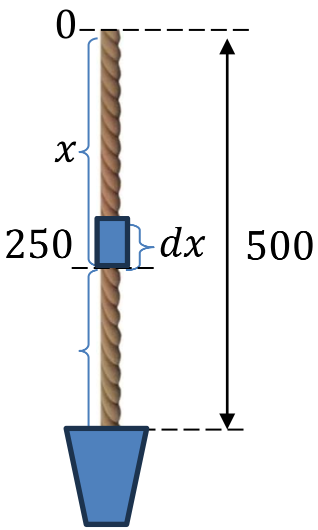
Limit

$dx \in [0,1]$

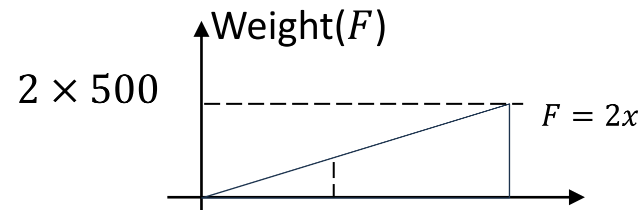
Volume

$\int_0^1 \pi[(5-ex)^2 - (5-e^x)^2]dx$

We have a cable that weighs 2 lbs/ft attached to a bucket filled with coal that weighs 750 lbs.
The bucket is initially at the bottom of a 500 ft mine shaft.
Determine the amount of work required to lift the bucket to the midpoint of the shaft.



- **Step 1** : plot a graph in the coordinate system (Cable weight vs length)
  - Set the top of the rope = 0



- **Step 2:** Slicing the cable by dx segment and consider a segment at location x (to be lifted by x)
  - Find the force at x =[weight of rope with length x]+[Bucket weight]
    - $F(x) = 2x + 750$
- **Step 3:** Find the work done by $F(x)$ over $[x, x + dx]$
  - $dW = F(x)dx = [2x + 750]dx$
- **Step 5.** Find the total work by integrating dW for dx ∈[portion of the cable lifted]
  - [portion of the cable lifted]= [0,250]
  - $W = \int_0^{250} [2x + 750]dx$

    $= [x^2 + 750x]_{250}^{500} = [500^2 - 250^2] + 750[500 - 250]$

    $= (500 + 250)(500 - 250) + 750[500 - 250] = 2 \cdot 750 \cdot 250$

    $= 375000$