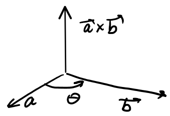


$\vec{c} = \vec{a}\times\vec{b}$

$\vec{c}\perp\vec{a}$ and $\vec{c}\perp\vec{b}$

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$|\vec{j}| = 1$

$|\vec{v}| = 5$

Math 251/221. WEEK in REVIEW 2. Fall 2024

1. Let $\mathbf{v} = 5\mathbf{j}$ and let $\mathbf{u}$ be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and the minimum values of the length of the vector $\mathbf{u}\times\mathbf{v}$. In what direction does $\mathbf{u}\times\mathbf{v}$ point?

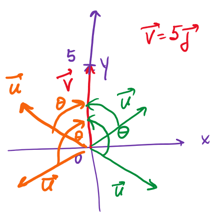
$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta = 5(3)\sin\theta = 15\sin\theta$

$|\vec{v}| = 5,\ |\vec{u}| = 3$

$|\vec{u}\times\vec{v}| = 15\sin\theta$ $(0\le\theta\le\pi)$ $0\le\sin\theta\le 1$

max length is when $\sin\theta = 1$ or $\theta = \frac{\pi}{2}$, $|\vec{u}\times\vec{v}| = 15$

min length is when $\sin\theta = 0$, or $\theta = 0, \pi$, $|\vec{u}\times\vec{v}| = 0$



if the vector $\vec{u}$ lies in the IVth or the 1st quadrant, and it is rotating in the counter clock wise direction, then the cross-product is directed upward.

if the vector $\vec{u}$ lies in the IInd or the IIIrd quadrants, and it is rotating in the counter clock wise direction, then the cross-product is directed down ward

2. Find two unit vectors orthogonal to both $\vec{p}$ $< 3, 2, 1 >$ and $\vec{q}$ $< -1, 1, 0 >$.

$\vec{p} = <3, 2, 1>$, $\vec{q} = <-1, 1, 0>$

vector orthogonal to both $\vec{p}$ and $\vec{q}$ is $\vec{p}\times\vec{q}$

$$\vec{p}\times\vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= \vec{i}(0-1) - \vec{j}(0-(-1)) + \vec{k}(3-(-2)) = -\vec{i} - \vec{j} + 5\vec{k}$$

$$|\vec{p}\times\vec{q}| = \sqrt{(-1)^2 + (-1)^2 + 5^2} = \sqrt{27}$$

$$\boxed{\vec{u}_1 = \frac{\vec{p}\times\vec{q}}{|\vec{p}\times\vec{q}|} = \frac{-\vec{i} - \vec{j} + 5\vec{k}}{\sqrt{27}}} \qquad \boxed{\vec{u}_2 = -\vec{u}_1 = \frac{\vec{i} + \vec{j} - 5\vec{k}}{\sqrt{27}}}$$

$\vec{x} = <x_1, x_2, x_3>$

$\vec{y} = <y_1, y_2, y_3>$

$$\vec{x}\times\vec{y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \vec{i}\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \vec{j}\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \vec{k}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= \vec{i}(x_2y_3 - x_3y_2) - \vec{j}(x_1y_3 - y_1x_3) + \vec{k}(x_1y_2 - x_2y_1)$$

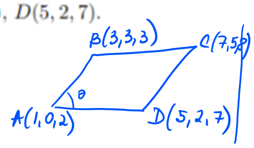
3. Find the area of the parallelogram with the vertices $A(1, 0, 2)$, $B(3, 3, 3)$, $C(7, 5, 8)$, $D(5, 2, 7)$.

[end] - [start]

$\vec{AB} = <3-1, 3-0, 3-2> = <2, 3, 1>$

$\vec{AD} = <5-1, 2-0, 7-2> = <4, 2, 5>$

$\vec{AB}$ and $\vec{AD}$ are not parallel



$A = |AB||AD|\sin\theta = |\vec{AB}\times\vec{AD}|$

$$\vec{AB}\times\vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}$$

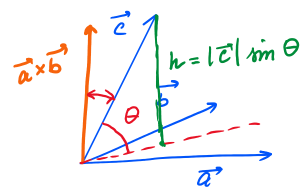
$$= \vec{i}(15-2) - \vec{j}(10-4) + \vec{k}(4-12)$$

$$= 13\vec{i} - 6\vec{j} - 8\vec{k}$$

$$A = |\vec{AB}\times\vec{AD}| = \sqrt{13^2 + (-6)^2 + (-8)^2} = \sqrt{169 + 36 + 64} = \boxed{\sqrt{269}}$$

$\vec{a}\cdot(\vec{b}\times\vec{c})$

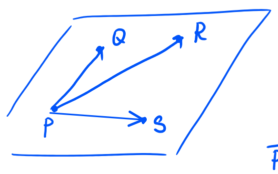
$|\vec{a}\cdot(\vec{b}\times\vec{c})| = |\vec{a}|\,|\vec{b}\times\vec{c}|\cdot\cos(\frac{\pi}{2}-\theta) = |\vec{a}|\,|\vec{b}\times\vec{c}|\sin\theta$



θ is an angle betwee $\vec{c}$ and the plane through $\vec{a}$ and $\vec{b}$

an angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is $\frac{\pi}{2}-\theta$

4. Use the scalar triple product to determine whether the points $P(1,0,1)$, $(Q(2,4,6))$, $R(3,-1,2)$, and $S(6,2,8)$ lie in the same plane.



case where $\theta=0$, and $\sin\theta=0$

if P, Q, R, and S lie in the same plane, then

$\vec{PQ}\cdot(\vec{PR}\times\vec{PS})=0$

$\vec{PQ} = \langle 2-1, 4, 6-1\rangle = \langle 1,4,5\rangle$

$\vec{PR} = \langle 3-1, -1, 2-1\rangle = \langle 2,-1,1\rangle$

$\vec{PS} = \langle 6-1, 2, 8-1\rangle = \langle 5,2,7\rangle$

$$\vec{PQ}\cdot(\vec{PR}\times\vec{PS}) = \begin{array}{ccc|cc} + & + & + - & - & - \\ 1 & 4 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & -1 \\ 5 & 2 & 7 & 5 & 2 \end{array}$$

$= -7+20+20+25-56-2$

$= 65-65 = 0$

the do lie in the same plane.

$\vec{a} = \langle a_1, a_2, a_3\rangle$

$\vec{b} = \langle b_1, b_2, b_3\rangle$

$\vec{c} = \langle c_1, c_2, c_3\rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{array}{ccc|cc} + & + & + - & - & - \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$

$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2$

$- a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$

5. Find the volume of the parallelepiped determined by the vectors $\mathbf{a} = 2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, $\mathbf{b} = \mathbf{i}-\mathbf{j}$, and $\mathbf{c} = 2\mathbf{i}+3\mathbf{k}$.

$V = (|h|)(A_{basis}) = |\vec{c}|\sin\theta\cdot|\vec{a}\times\vec{b}| = |\vec{c}\cdot(\vec{a}\times\vec{b})|$

the absolute value.

$A_{basis} = |\vec{a}\times\vec{b}|$

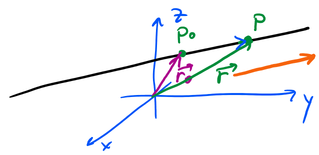
$|h| = |\vec{c}|\sin\theta$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{array}{ccc|cc} + & + & + - & - & - \\ 2 & 3 & -2 & 2 & 3 \\ 1 & -1 & 0 & 1 & -1 \\ 2 & 0 & 3 & 2 & 0 \end{array}$$

$= -6-4-9 = -19$

Volume $= |-19| = \boxed{19}$

a line in $\mathbb{R}^3$.
a point on the line $P_0(x_0, y_0, z_0)$ and a vector parallel to the line.



$\overrightarrow{P_0P} \parallel \vec{v}$

$\overrightarrow{P_0P} = t\vec{v}$

$\vec{r} = \vec{r_0} + \overrightarrow{P_0P}$, $\vec{r_0} = \langle x_0, y_0, z_0\rangle$

$\vec{r} = \vec{r_0} + t\vec{v}$

$\boxed{\vec{r} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle}$ vector equation

point parallel vector

$\vec{r} = \langle x, y, z\rangle$

$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$

$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$

match up the components:

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$ – parametric equations.

point parallel vector

solve parametric equations for t

$t = \dfrac{x - x_0}{a}$, $t = \dfrac{y - y_0}{b}$, $t = \dfrac{z - z_0}{c}$

$\boxed{\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$ symmetric equations.

6. Find a **vector** equation and **parametric** equations for the line passing through the point $P(-3, 2, 1)$ and parallel to the vector $\mathbf{v} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

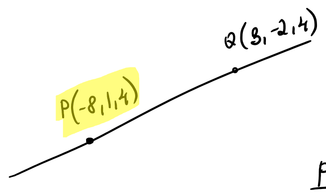
x_0 y_0 z_0 ; a b c ; point ; parallel vector

vector equation $\boxed{\vec{r} = \langle -3, 2, 1\rangle + t\langle -2, 1, -3\rangle}$

$\langle x, y, z\rangle = \langle -3 - 2t, 2 + t, 1 - 3t\rangle$

$$\begin{bmatrix} x = -3 - 2t \\ y = 2 + t \\ z = 1 - 3t \end{bmatrix}$$ parametric equations.

7. Find **parametric** and **symmetric** equations for the line through the points $P(-8, 1, 4)$ and $Q(3, -2, 4)$. Find the points in which the line intersects the coordinate planes.



the vector $\overrightarrow{PQ}$ is parallel to the line

$\overrightarrow{PQ} = \langle 3 - (-8), -2 - 1, 4 - 4\rangle$

$= \langle 11, -3, 0\rangle$

parametric equations: $\begin{cases} x = -8 + t(11) \\ y = 1 + t(-3) \\ z = 4 + t(0) \end{cases} \Rightarrow \begin{bmatrix} x = -8 + 11t \\ y = 1 - 3t \\ z = 4 \end{bmatrix}$

the line lies in the plane $z = 4$.

symmetric equations: $\boxed{\dfrac{x+8}{11} = \dfrac{y-1}{-3},\ z = 4}$

Points in which the line intersects coordinate planes. The line lies in the plane $z = 4$, thus it will never intersect the xy-plane. xy-plane $(z=0)$. The line lies in the plane $z=4$, thus it will never intersect the xy-plane

xz-plane $(y=0)$. Find t such that $y = 0$

$y = 1 - 3t = 0$, solve for t

$t = \frac{1}{3}$

Find the corresponding $x\left(\frac{1}{3}\right) = -8 + \dfrac{11}{3} = \dfrac{-24 + 11}{3} = -\dfrac{13}{3}$

Point of intersection is $\left(-\frac{13}{3}, 0, 4\right)$

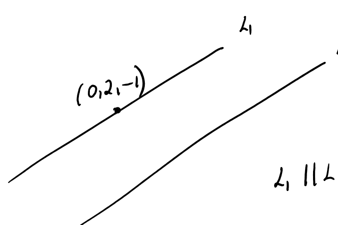
yz-plane $(x=0)$ Find t such that $x = 0$ or $x = -8 + 11t = 0$ solve for t

$t = \frac{8}{11}$

Find the corresponding $y\left(\frac{8}{11}\right) = 1 - 3\left(\frac{8}{11}\right) = 1 - \dfrac{24}{11} = \dfrac{11 - 24}{11} = -\dfrac{13}{11}$

Point of intersection is $\left(0, -\frac{13}{11}, 4\right)$

8. Find symmetric equations for the line that passes through the point $(0, 2, -1)$ and is parallel to the line with parametric equations $x = -1 + 2t$, $y = 6 - 3t$, and $z = 5 + 9t$.



L_2: $\begin{bmatrix} x = -1 + 2t \\ y = 6 - 3t \\ z = 5 + 9t \end{bmatrix}$

$L_1 \parallel L_2$

The vector parallel to L_2 is $\langle 2, -3, 9\rangle$

$\boxed{\dfrac{x - 0}{2} = \dfrac{y - 2}{-3} = \dfrac{z + 1}{9}}$

$$\boxed{\frac{x-0}{2} = \frac{y-2}{-3} = \frac{z+1}{9}}$$

9. Determine whether the lines L_1 and L_2 are parallel, skew (do not intersect and are not parallel), or intersecting. If they intersect, find the point of intersection.

(a) $L_1: x = 2 - 6t, y = 9t, z = 1 - 3t,$ $L_2: x = 1 + 2s, y = 4 - 3s, z = s.$

$\vec{v}_1 = \langle -6, 9, -3 \rangle$ $\vec{v}_2 = \langle 2, -3, 1 \rangle$

$\vec{v}_1 = -3\vec{v}_2$, the vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel and so are the lines L_1 and L_2

$L_1 \parallel L_2$ $\boxed{\text{(parallel)}}$

(b) $L_1: x = 3 + 2t, y = 4 - t, z = 1 + 3t,$ $L_2: x = 1 + 4s, y = 3 - 2s, z = 4 + 5s.$

$\vec{v}_1 = \langle 2, -1, 3 \rangle$ $\vec{v}_2 = \langle 4, -2, 5 \rangle$

$\vec{v}_1$ and $\vec{v}_2$ are not parallel

Find the point of intersection.

match up equations for x, y, and z for the lines L_1 and L_2

$$\begin{cases} x: 3+2t = 1+4s \\ y: 4-t = 3-2s \\ z: \cancel{1+3t = 4+5s} \end{cases} \qquad \begin{cases} 2t-4s=-2 \\ 2(t-2s) = 1(2) \\ 3t-5s=-3 \end{cases}$$

2nd eqn times 2 is

$$\begin{array}{r} 2t-4s=2 \\ - \; 2t-4s=-2 \leftarrow \text{first equation} \\ \hline 0 \overset{?}{=} 4 \text{ (impossible)} \end{array}$$

$\boxed{\text{skew lines}}$

(c) $L_1: \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} = t$ $L_2: \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7} = s$

$\vec{v}_1 = \langle 1, -2, -3 \rangle$ $\vec{v}_2 = \langle 1, 3, -7 \rangle$ these vectors are not parallel

Find the point of intersection.

write parametric equations for L_1 and L_2.

$$L_1: \begin{cases} x = 2+t \\ y = 3-2t \\ z = 1-3t \end{cases} \qquad L_2: \begin{cases} x = 3+s \\ y = -4+3s \\ z = 2-7s \end{cases}$$

plug in $t=2$

$x = 2+2 = 4$

$y = 3-4 = -1$

$z = 1-3(2) = -5$

match up the equations:

$$\begin{cases} x: 2+t = 3+s \\ y: 3-2t = -4+3s \\ z: 1-3t = 2-7s \end{cases} \quad \text{or} \quad \begin{cases} t = 1+s \\ \cancel{+2t = +7-3s} \\ +3t = -1+7s \end{cases}$$

$$\begin{cases} 2t = 2(1+s) \\ 2t = 7-3s \end{cases}$$

$$\begin{array}{r} 2t = 2+2s \\ - \; 2t = 7-3s \\ \hline 0 = 2-7+2s+3s \end{array}$$

$0 = -5+5s$ or $\boxed{s=1}$ $\boxed{t = 1+s = 2}$

Check if $s=1$ and $t=2$ satisfy the second equation:

$2\overset{2}{\cancel{t}} = 7-3\overset{1}{\cancel{s}}$

$2(2) = 7-3$

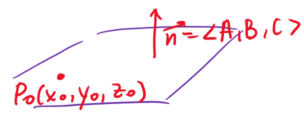
$4 = 4$

The lines are intersecting.

To find the point of intersection, plug $t=2$ into the parametric equations of L_1

or plug $s=1$ into the param equations of L_2.

$$L_2: \begin{cases} x = 3 + \cancel{s}\,1 \\ y = -4 + 3\cancel{s}\,1 \\ z = 2 - 7\cancel{s}\,1 \end{cases} \Rightarrow \text{Point } \boxed{(4, -1, -5)}$$



$\vec{n}$ is a normal (perpendicular) vector to the plane.
$P_0(x_0, y_0, z_0)$ is a point in the plane.

$$\boxed{A(x-x_0)+B(y-y_0)+C(z-z_0)=0}$$

10. Find an equation of the plane through the point $(\underset{y_0}{\overset{x_0}{3}}, 5, \underset{z_0}{3})$ with the normal vector $\mathbf{n} = \langle \underset{A}{2}, \underset{B}{1}, \underset{C}{-1} \rangle$. Find the intercepts and sketch the plane.

$$\boxed{2(x-3)+1(y-5)-1(z-3)=0}$$

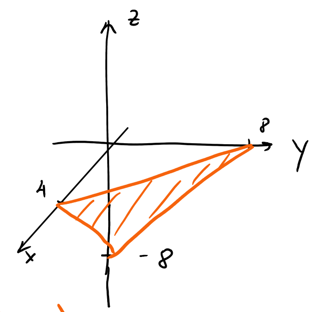
$2x-6+y-5-z+3=0$

$2x+y-z=8$

x-axis $(y=z=0)$ $\quad 2x+\cancel{y}-\cancel{z}=8 \Rightarrow 2x=8 \Rightarrow x=4 \quad \boxed{(4,0,0)}$

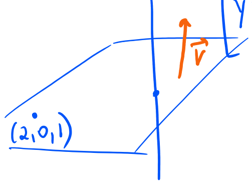
y-axis $(x=z=0)$ $\quad \cancel{2x}+y-\cancel{z}=8 \Rightarrow y=8, \quad \boxed{(0,8,0)}$

z-axis $(x=y=0)$ $\quad \cancel{2x}+\cancel{y}-z=8 \Rightarrow -z=8$ or $z=-8 \quad \boxed{(0,0,-8)}$



11. Find an equation of the plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$.

$L: \begin{cases} x=3t \\ y=2-t \\ z=3+4t \end{cases}$

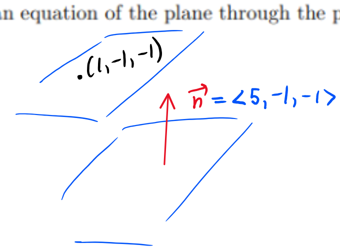


$\vec{v} = \langle 3,-1,4\rangle$ (vector parallel to the line L)

$\vec{n} = \vec{v} \perp$ to the plane

$$\boxed{3(x-2)-1(y-0)+4(z-1)=0}$$

12. Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.



The parallel planes share the normal vector

$$\boxed{5(x-1)-1(y+1)-1(z+1)=0}$$

16. Find the point at which the line $[x = t - 1,\ y = 1 + 2t,\ z = 3 - t]$ intersects the plane $2x - y + 2z = 5$.

$$2\underbrace{(t-1)}_{x}-\underbrace{(1+2t)}_{y}+2\underbrace{(3-t)}_{z}=5$$

solve for t: $\quad \cancel{2t}-2-1-\cancel{2t}+6-2t=5$

$2t=-2 \Rightarrow t=-1$

point of intersection $\quad x=\overset{-1}{\cancel{t}}-1 \Rightarrow x=-1-1=-2$

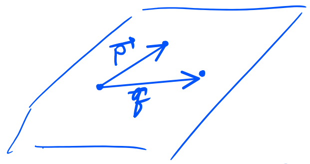
$y=1+2\overset{-1}{\cancel{t}} \Rightarrow y=1-2=-1$

$z=3-\overset{-1}{\cancel{t}} \Rightarrow z=3-(-1)=4$

$$\boxed{(-2,-1,4)}$$

[start]

14. Find an equation of the plane passing through the points $(-1, 1, -1)$, $(1, -1, 0)$, $(1, 0, 1)$.



$\vec{n} = \vec{p} \times \vec{q}$

$\vec{p} = \langle 1-(-1),\ -1-1,\ 0-(-1) \rangle = \langle 2, -2, 1 \rangle$

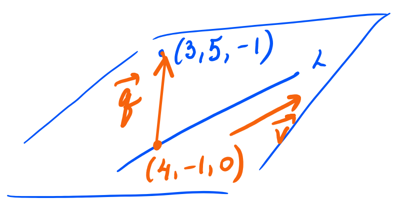
$\vec{q} = \langle 1-(-1),\ 0-1,\ 1-(-1) \rangle = \langle 2, -1, 2 \rangle$

$$\vec{n} = \vec{p} \times \vec{q} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ 2 & -1 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -2 \\ 2 & -1 \end{vmatrix}$$

$$= -3\vec{i} - 2\vec{j} + 2\vec{k}$$

equation: $\boxed{-3(x+1) - 2(y-1) + 2(z+1) = 0}$

15. Find an equation of the plane that passes through the point $(3, 5, -1)$ and contains the line $x = 4 - t$, $y = 2t - 1$, $z = -3t$.



L: $x = 4 - t$

$y = 2t - 1$

$z = -3t + 0$

$\vec{v} = \langle -1, 2, -3 \rangle$ parallel to L

$P(4, -1, 0)$

$\vec{q} = \langle 3-4,\ 5-(-1),\ -1-0 \rangle = \langle -1, 6, -1 \rangle$

$$\vec{n} = \vec{q} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ -1 & 6 & -1 \end{vmatrix} = 16\vec{i} + 2\vec{j} - 4\vec{k}$$

$$\boxed{16(x-3) + 2(y-5) - 4(z+1) = 0}$$