1. Evaluate the integral $\iiint_E (xy + z^2)\,dV$, where $E = \overset{x}{[0,2]} \times \overset{y}{[0,1]} \times \overset{z}{[0,3]}$.

$$\int_0^1\int_0^3\int_0^2 (xy+z^2)\,dx\,dz\,dy = \int_0^1\int_0^3 \left(\frac{x^2}{2}y + z^2x\right)\Big|_0^2 dz\,dy$$

$$= \int_0^1\int_0^3 (2y + 2z^2)\,dz\,dy = \int_0^1 \left(2yz + \frac{2z^3}{3}\right)\Big|_0^3 dy = \int_0^1 \left(6y + \frac{27}{2}\right) dy$$

$$= \left(3y^2 + \frac{27}{2}y\right)\Big|_0^1 = 3 + \frac{27}{2} = \frac{33}{2}$$

2. Evaluate the iterated integral $\int_0^2 \int_0^{z^2} \int_0^{y-z} (2x - y)\, dx\, dy\, dz = \int_0^2 \int_0^{z^2} \left(x^2 - xy\right)\Big|_0^{y-z} dy\, dz$

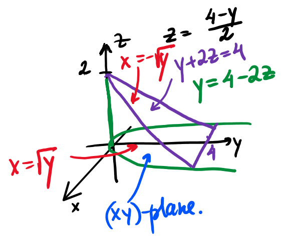
$$= \int_0^2 \int_0^{z^2} \left[(y-z)^2 - (y-z)y\right] dy\, dz = \int_0^2 \int_0^{z^2} \left[\cancel{y^2} - 2yz + z^2 - \cancel{y^2} + yz\right] dy\, dz$$

$$= \int_0^2 \int_0^{z^2} \left(z^2 - yz\right) dy\, dz = \int_0^2 \left(yz^2 - \frac{y^2}{2} z\right)\Big|_0^{z^2} dz = \int_0^2 \left(z^4 - \frac{z^5}{2}\right) dz$$

$$= \left(\frac{z^5}{5} - \frac{z^6}{12}\right)\Big|_0^2 = \frac{32}{5} - \frac{64}{12} = \frac{32}{5} - \frac{16}{3} = \frac{16}{15}$$

3. Express the integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in six different ways, where E is the solid bounded by the surfaces $\underbrace{y=x^2}_{\text{parabolic cylinder}}$, $\underbrace{z=0}_{(xy)\text{-plane}}$, $\underbrace{y+2z=4}_{\text{plane}}$. parallel to the x-axis.

$y=x^2 \Rightarrow x=\pm\sqrt{y}$



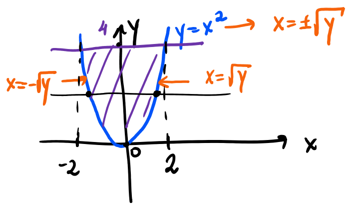
$y+2z=4$

$z=0 \Rightarrow y=4 \rightarrow (0,4,0)$

$y=0 \Rightarrow 2z=4$ or $z=2 \Rightarrow (0,0,2)$

Projections

(xy)-plane

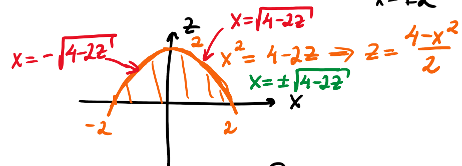


(xz)-plane.

$y=x^2$, $y=4-2z \rightarrow x^2=4-2z$

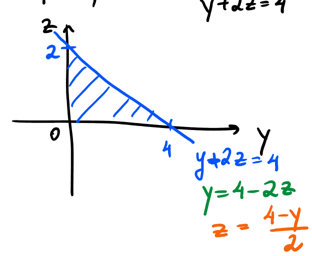
$x=0 \Rightarrow z=2$

$z=0 \Rightarrow x^2=4$, $x=\pm 2$



(yz)-plane

$y+2z=4$



dzdxdy: $0 \le z \le \frac{4-y}{2}$, $-\sqrt{y} \le x \le \sqrt{y}$, $0 \le y \le 4$

dzdydx: $0 \le z \le \frac{4-y}{2}$, $x^2 \le y \le 4$, $-2 \le x \le 2$

dydzdx: cylinder $\le y \le$ plane; $x^2 \le y \le 4-2z$, $0 \le z \le \frac{4-x^2}{2}$, $-2 \le x \le 2$

dydxdz: $x^2 \le y \le 4-2z$, $-\sqrt{4-2z} \le x \le \sqrt{4-2z}$, $0 \le z \le 2$

dxdydz: [back half of the cylinder] $\le x \le$ [front half of the cylinder]; $-\sqrt{y} \le x \le \sqrt{y}$, $0 \le y \le 4-2z$, $0 \le z \le 2$

dxdzdy: $-\sqrt{y} \le x \le \sqrt{y}$, $0 \le z \le \frac{4-y}{2}$, $0 \le y \le 4$

$$\iiint_V f\,dV = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{\frac{4-y}{2}} f\,dz\,dx\,dy = \int_{-2}^{2} \int_{x^2}^{4} \int_0^{\frac{4-y}{2}} f\,dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_0^{\frac{4-x^2}{2}} \int_{x^2}^{4-2z} f\,dy\,dz\,dx = \int_0^2 \int_{-\sqrt{4-2z}}^{\sqrt{4-2z}} \int_{x^2}^{4-2z} f\,dy\,dx\,dz$$

$$= \int_0^2 \int_0^{4-2z} \int_{-\sqrt{y}}^{\sqrt{y}} f\,dx\,dy\,dz = \int_0^4 \int_0^{\frac{4-y}{2}} \int_{-\sqrt{y}}^{\sqrt{y}} f\,dx\,dz\,dy$$

$$\boxed{\int e^{ax}\,dx = \frac{1}{a}e^{ax}+C}$$

4. Evaluate the triple integral

(a) $\iiint_E e^{z/y}\,dV$, where $E=\{(x,y,z): 0\le \overset{(3)}{y}\le 1, y\le \overset{(2)}{x}\le 1, 0\le \overset{(1)}{z}\le xy\}$

$$= \int_0^1\int_y^1\int_0^{xy} e^{\frac{z}{y}}\,dz\,dx\,dy = \int_0^1\int_y^1 e^{\frac{z}{y}}\,y\Big|_0^{xy}\,dx\,dy = \int_0^1\int_y^1 \left(ye^{x}-ye^{0}\right)dx\,dy$$

$$= \int_0^1\int_y^1 \left(ye^{x}-y\right)dx\,dy = \int_0^1 \left(ye^{x}-yx\right)\Big|_y^1 dy = \int_0^1 \left(\underbrace{ye^{y}}_{\text{by parts}}-y^2\right)dy$$

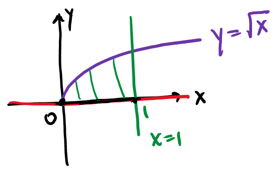
	D	I
+	y	e^y
−	1	e^y
	0	e^y

$$= \left(ye^{y}-e^{y}-\frac{y^3}{3}\right)\Big|_0^1 = e^1-e^1-\frac{1}{3}+1 = 1-\frac{1}{3} = \frac{2}{3}$$

$$\left[(xy)\text{ plane}\right] \le z \le \left[\begin{matrix}\text{plane}\\ z=1+x+y\end{matrix}\right] \Rightarrow \boxed{0 \le z \le 1+x+y}$$

(b) $\displaystyle\iiint_E 6xy\,dV$, where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $\underbrace{y = \sqrt{x},\ y = 0, \text{ and } x = 1}_{\text{projection onto the }(xy)\text{-plane.}}$.

[$y=0$ is the x-axis]



$[x\text{-axis}] \le y \le [\text{parabola}]$

$0 \le y \le \sqrt{x}$

$0 \le x \le 1$

$$= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

$$= 6\int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} xy\,dz\,dy\,dx = 6\int_0^1 \int_0^{\sqrt{x}} xyz\Big|_0^{1+x+y} dy\,dx$$

$$= 6\int_0^1 \int_0^{\sqrt{x}} xy(1+x+y)\,dy\,dx = 6\int_0^1 \int_0^{\sqrt{x}} (xy + x^2y + xy^2)\,dy\,dx$$

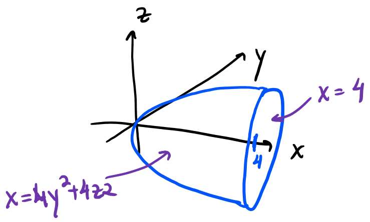
$$= 6\int_0^1 \left[x\frac{y^2}{2} + x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_0^{\sqrt{x}} dx$$

$$= 6\int_0^1 \left(\frac{x^2}{2} + \frac{x^3}{2} + \frac{x\cdot x^{3/2}}{3}\right) dx = 6\left[\frac{x^3}{6} + \frac{x^4}{8}\right]_0^1 + 2\int_0^1 x^{5/2}\,dx$$

$$= 6\left(\frac{1}{6} + \frac{1}{8}\right) + 2\,\frac{x^{7/2}}{7/2}\Big|_0^1 = 1 + \frac{3}{4} + \frac{4}{7} = \ldots$$

(c) $\iiint_E x\,dV$, where E is bounded by a paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

$$\int_a^b \int_c^d f(x)\,g(y)\,dx\,dy = \int_a^b g(y)\,dy \cdot \int_c^d f(x)\,dx$$



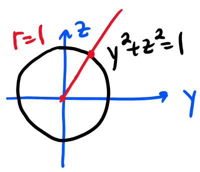
[paraboloid] $\le x \le$ [plane $x=4$]

$4y^2+4z^2 \le x \le 4$

$\boxed{4r^2 \le x \le 4}$

projection onto yz-plane.

$\left.\begin{array}{l} x=4y^2+4z^2 \\ x=4 \end{array}\right\} \Rightarrow \begin{array}{l} 4=4y^2+4z^2 \\ y^2+z^2=1 \end{array}$



polar coordinates for y and z

$y = r\cos\theta$

$z = r\sin\theta$

$y^2+z^2 = r^2$

$\boxed{\begin{array}{l} 0\le r\le 1 \\ 0 \le \theta \le 2\pi \end{array}}$

$dy\,dz = r\,dr\,d\theta$

$$\int_0^{2\pi}\int_0^1\int_{4r^2}^4 x r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\,\frac{x^2}{2}\Big|_{4r^2}^{4}\,dr\,d\theta$$

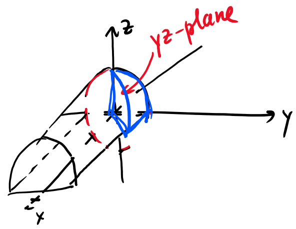
$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r\left(16-16r^4\right)dr\,d\theta$$

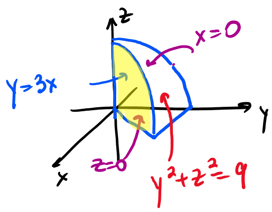
$$= \frac{1}{2}\underbrace{\int_0^{2\pi} d\theta}_{2\pi} \cdot \int_0^1 \left(16r - 16r^5\right)dr$$

$$= 16\pi\left(\frac{r^2}{2} - \frac{r^6}{6}\right)\Big|_0^1$$

$$= 8\pi\left(1 - \frac{1}{3}\right) = \frac{16\pi}{3}$$

(d) $\iiint_E z\,dV$, where E is bounded by the cylinder $y^2+z^2=9$ and the planes $\overbrace{x=0}^{yz\text{-plane}}$, $y=3x$, and $\underbrace{z=0}_{xy\text{-plane.}}$ in the first octant.

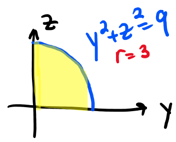




$[yz\text{-plane}] \le x \le [\text{plane } y=3x]$ $\quad x=\frac{y}{3}$

$\boxed{0 \le x \le \frac{y}{3}}$

$\boxed{0 \le x \le \frac{1}{3} r\cos\theta}$



polar coordinates for y and z

$y = r\cos\theta$

$z = r\sin\theta$

$dy\,dz = r\,dr\,d\theta$

$y^2+z^2=r^2$

$\boxed{\begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le \frac{\pi}{2} \end{array}}$

$$\iiint_E \underset{r\sin\theta}{z}\,dV = \int_0^{\pi/2}\int_0^3\int_0^{\frac{1}{3}r\cos\theta} \underbrace{r\sin\theta}_{z}\; r\,dx\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^3 r^2\sin\theta \times \int_0^{\frac{1}{3}r\cos\theta} \quad dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^3 r^2\sin\theta\left(\frac{1}{3}r\cos\theta\right) dr\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\int_0^3 (r^3)(\sin\theta\cos\theta)\,dr\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \int_0^3 r^3\,dr$$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

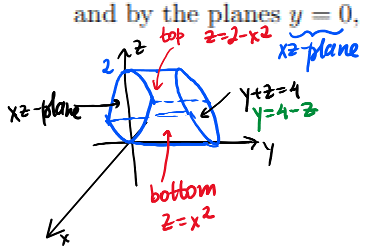
$\theta = 0 \rightarrow u = \sin 0 = 0$

$\theta = \frac{\pi}{2} \rightarrow u = \sin\frac{\pi}{2} = 1$

$$= \frac{1}{3}\int_0^1 u\,du \cdot \frac{r^4}{4}\Big|_0^3$$

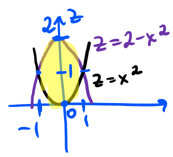
$$= \frac{1}{3}\,\frac{u^2}{2}\Big|_0^1 \cdot \frac{81}{4} = \frac{27}{8}$$

5. Use a triple integral to find the volume of the region bounded by the parabolic cylinders $z = x^2$, $z = 2 - x^2$ and by the planes $y = 0$, $y + z = 4$.



$0 \le y \le 4-z$

$$V = \iiint_E 1\,dV = \int_{-1}^{1}\int_{x^2}^{2-x^2}\int_0^{4-z} dy\,dz\,dx$$



$dz\,dx$

$x^2 \le z \le 2-x^2$

$-1 \le x \le 1$

$$= \int_{-1}^{1}\int_{x^2}^{2-x^2} y\Big|_0^{4-z}\,dz\,dx$$

$$= \int_{-1}^{1}\int_{x^2}^{2-x^2} (4-z)\,dz\,dx = \int_{-1}^{1}\left(4z - \frac{z^2}{2}\right)_{x^2}^{2-x^2} dx$$

$2-x^2 = x^2$

$2 = 2x^2 \Rightarrow x = \pm 1$

$$= \int_{-1}^{1}\left(4(2-x^2) - \frac{(2-x^2)^2}{2} - 4x^2 + \frac{x^4}{2}\right) dx$$

$$= \int_{-1}^{1}\left(8 - 4x^2 - \frac{4 - 4x^2 + \cancel{x^4}}{2} - 4x^2 + \cancel{\frac{x^4}{2}}\right) dx$$

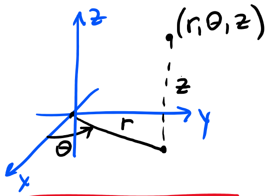
$$= \int_{-1}^{1}\left(8 - 4x^2 - 2 + 2x^2 - 4x^2\right) dx$$

$$= \int_{-1}^{1}\left(6 - 6x^2\right) dx = \left(6x - \frac{6x^3}{3}\right)_{-1}^{1} = 6(2) - 2(1+1)$$

$$= 12 - 4 = 8.$$

cylindrical

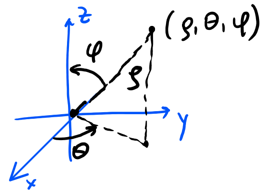
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



spherical

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

$x^2+y^2+z^2 = \rho^2$



$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

6. Write the equation $\boxed{x^2 + y^2 + z^2 = 4y}$ in cylindrical and spherical coordinates.

cylindrical

$r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 4r\sin\theta$

$r^2(\cos^2\theta + \sin^2\theta) + z^2 = 4r\sin\theta$

$\boxed{r^2 + z^2 = 4r\sin\theta}$

$\underbrace{x^2+y^2+z^2}_{} = 4y$

$\rho^{\cancel{2}} = 4\cancel{\rho}\sin\theta\sin\varphi$

$\boxed{\rho = 4\sin\theta\sin\varphi}$

7. Convert the integral

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy$$

to an integral in cylindrical coordinates, but don't evaluate it.

$x^2+y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

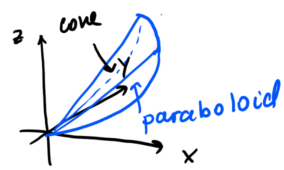
$dz\,dx\,dy = r\,dz\,dr\,d\theta$

$x^2+y^2 \le z \le \sqrt{x^2+y^2}$

$z = x^2+y^2 \leftarrow$ paraboloid circular

$z = \sqrt{x^2+y^2}$ or $z^2 = x^2+y^2$ top half of the cone.

The solid lies in the first octant.



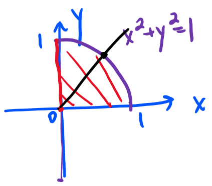
projection (xy)-plane

$0 \le x \le \sqrt{1-y^2} \Rightarrow x = 0 -$ y-axis

$x = \sqrt{1-y^2} \longrightarrow x^2 = 1-y^2$

$x^2+y^2 = 1$

$0 \le y \le 1$



$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$

$x^2+y^2 \le z \le \sqrt{x^2+y^2}$

$r^2 \le z \le r$

$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$

$xyz = \underbrace{r\cos\theta}_{x} \; \underbrace{r\sin\theta}_{y} \; z = r^2 z \cos\theta \sin\theta$

$dz\,dx\,dy = r\,dz\,dr\,d\theta$

$$\int_0^{\pi/2} \int_0^1 \int_{r^2}^{r} \underbrace{r^2 z \cos\theta \sin\theta}_{xyz} \; r \underbrace{dz\,dr\,d\theta}_{dx\,dy\,dy}$$