Week 14

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TEXAS A&M UNIVERSITY
Math Learning Center

Math 151
Week-In-Review 14

5.4, 5.5
Todd Schrader

Problem Statements

1. Find the general indefinite integral.

$2 \cdot x^{-3} + 3x^{-1} + x^{2/5}$

(a) $\int \left(3x^2 + 8x - 5 + \frac{2}{x^3} + \frac{3}{x} + \sqrt[5]{x^2}\right) dx$

$3 \cdot \frac{1}{3}x^3 + 8 \cdot \frac{1}{2}x^2 - 5x + 2\left(\frac{-1}{2}x^{-2}\right) + 3\ln|x| + \frac{5}{7}x^{7/5} + C$

$2^{3x} = (2^3)^x = 8^x$ $\quad 7^{x+5} = 7^x \cdot 7^5 = 7^5 \cdot 7^x$

(b) $\int \left(11e^x - 3^x + 2 \cdot 5^x + 2^{3x} - 7^{x+5}\right) dx$

$11e^x - \frac{3^x}{\ln(3)} + 2 \cdot \frac{5^x}{\ln(5)} + \frac{8^x}{\ln(8)} - 7^5 \cdot \frac{7^x}{\ln(7)} + C$

(c) $\int \left(2\sin x + 4\cos x - 6\sec^2 x + 8\sec x \tan x\right) dx$

$2(-\cos x) + 4(\sin x) - 6\tan(x) + 8\sec(x) + C$

$7 \cdot \frac{1}{x^2+1} - 9 \cdot \frac{1}{\sqrt{1-x^2}}$

(d) $\int \left(3\csc^2 x - 5\csc x \cot x + \frac{7}{x^2+1} - \frac{9}{\sqrt{1-x^2}}\right) dx$

$3(-\cot(x)) - 5(-\csc x) + 7\arctan(x) - 9\arcsin(x) + C$

$V =$ Volume

2. A water balloon is being filled at a constant rate of 100 cm^3/s. $V'(t) = 100$

   (a) Find an equation representing the volume of water in the balloon after t seconds.

   $V = \int V'(t)\,dt = \int 100\,dt = 100t + C$ $\boxed{V(t) = 100t}$

   Assume $V(0) = 0$ $100(0) + C = 0$ $C = 0$

   (b) How much water is added to the balloon in the interval from $t = 1$ to $t = 3$?

   $V(3) - V(1) = 100(3) - 100(1) = 200$

   $\int_1^3 V'(t)\,dt = V(t)\Big|_1^3 = V(3) - V(1) = 200$

   (c) Suppose instead that the balloon immediately springs a leak, and water begins to drain from the balloon at a rate of $0.5V$. (That is, more water in the balloon results in water leaking out of the balloon faster). Write a *differential equation* representing the total rate of change of the volume of water in the balloon.

   Volume

   $V' = 100 - 0.5V$

   (d) Verify $V = 200 - 200e^{-0.5t}$ is a solution to the differential equation.

   $100e^{-0.5t} = 100 - 0.5(200 - 200e^{-0.5t})$

   $= 100 - 100 + 100e^{-0.5t} = 100e^{-0.5t}$ ✓

   (e) Find $V'(t)$.

   $V'(t) = -200 \cdot e^{-0.5t} \cdot (-0.5) = 100e^{-0.5t}$

   (f) What is the net change in the amount of water in the balloon in the interval from $t = 1$ to $t = 3$?

   $\Delta V = \int_1^3 V'(t)\,dt = V(t)\Big|_1^3 = V(3) - V(1)$

   $V(3) = 200 - 200e^{-3/2}$

   $V(1) = 200 - 200e^{-1/2}$

   $\boxed{\Delta V = -200e^{-3/2} + 200e^{-1/2}} \approx 76.68\ cm^3$

   (g) What is the net change in the amount of water in the balloon in the interval from $t = 101$ to $t = 103$?

   $\Delta V = \int_{101}^{103} V'(t)\,dt = V(103) - V(101) = \boxed{-200e^{-103/2} + 200e^{-101/2}}$

   ≈ 0

3. The velocity function for a particle moving along a straight line is given by $v(t) = 3t^2 - 3t - 6$. On the interval $1 \le t \le 3$, determine

(a) The displacement of the particle.

Displacement $= \int_1^3 v(t)\,dt = \int_1^3 (3t^2 - 3t - 6)\,dt$

$= \left[t^3 - \frac{3}{2}t^2 - 6t\right]_1^3$

$\int v(t)\,dt = t^3 - \frac{3}{2}t^2 - 6t$

$= \left[3^3 - \frac{3}{2}(3^2) - 6(3)\right] - \left[1^3 - \frac{3}{2}(1^2) - 6(1)\right]$

$S(3) = \frac{-9}{2}$

$S(1) = \frac{-13}{2}$

$= \left(27 - \frac{27}{2} - 18\right) - \left(1 - \frac{3}{2} - 6\right)$

$= \frac{-9}{2} - \left(\frac{-13}{2}\right) = \frac{4}{2} = \boxed{2}$

(b) The total distance traveled by the particle.

Distance: $\int_1^3 |v(t)|\,dt$

$s(t) = t^3 - \frac{3}{2}t^2 - 6t$

$s(1) = \frac{-13}{2}$

$s(2) = 2^3 - \frac{3}{2}(2^2) - 6(2) = -10$

$s(3) = -\frac{9}{2}$

Relevant: $v(t) = 0$?

$3t^2 - 3t - 6 = 0$

$3(t^2 - t - 2) = 0$

$3(t-2)(t+1) = 0$

$t = 2 \quad t = -1$

$|s(2) - s(1)| = \left|-10 - \left(\frac{-13}{2}\right)\right| = \frac{7}{2}$

$|s(3) - s(2)| = \left|\frac{-9}{2} - (-10)\right| = \frac{11}{2}$

Distance: $+ = \frac{18}{2} = \boxed{9}$

Could $\int_1^2 |v(t)|\,dt + \int_2^3 |v(t)|\,dt$

4. If $h(t)$ represents a person's heart rate in beats per minute t minutes into a workout, what does $\int_0^{30} h(t)\, dt$ represent?

$\int_0^{30} h(t)\, dt$ represents the total number of heart beats in the first 30 minutes of the workout.

5. If $s(m)$ represents the slope of a trail at a distnce of m miles from the trailhead, what does $\int_2^4 s(m)\, dm$ represent?

Rate of change of elevation per mile from trailhead

$\int_2^4 s(m)\, dm$ represents the net change in elevation in the two mile interval between 2 miles from the trailhead to 4 miles from the trailhead.

6. Evaluate

(a) $\frac{d}{dx}\left[(x^2+1)^{100}\right]$

Chain Rule!

$$= 100\,(x^2+1)^{99}\cdot(2x)$$

u - Substitution

(b) $\int 100(x^2+1)^{99}\cdot 2x\,dx \quad = (x^2+1)^{100} + C$

$u = x^2+1$

$du = 2x\,dx$

$$\int 100\,u^{99}\,du = 100\,\frac{1}{100}\,u^{100} + C$$

$$= u^{100} + C = \boxed{(x^2+1)^{100} + C}$$

(c) $\frac{1}{4}\int 4x^3(x^4+1)^{99}\,dx$

$u = x^4+1$

$du = 4x^3\,dx$

$dx = \frac{du}{4x^3}$

$$\frac{1}{4}\int u^{99}\,du = \frac{1}{4}\cdot\frac{1}{100}\,u^{100} + C = \boxed{\frac{1}{400}\,(x^4+1)^{100} + C}$$

(d) $\frac{1}{4}\int x^4\,4x^3\,dx$

$u = x^4$

$du = 4x^3\,dx$

$$\frac{1}{4}\int u\,du = \frac{1}{4}\cdot\frac{1}{2}\,u^2 + C = \frac{1}{8}\,(x^4)^2 + C = \boxed{\frac{1}{8}\,x^8 + C}$$

7. Evaluate

(a) $\frac{1}{2}\int \sqrt{2x+1}\, dx$

$u = 2x+1$

$du = 2\,dx$

$$= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \boxed{\frac{1}{3}(2x+1)^{3/2} + C}$$

(b) $\int \sqrt{2x+1}\, dx$

$u = (2x+1)^{1/2}$

$du = \frac{1}{2}(2x+1)^{-1/2}\,2\,dx$

$dx = (2x+1)^{1/2}\,du$

$$\int u\,(2x+1)^{1/2}\,du$$

$$\int u\cdot u\,du = \int u^2\,du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}(2x+1)^{3/2} + C}$$

(c) $\int 4\cos(x)\sqrt{\sin(x)+1}\, dx$

$u = \sin(x)+1$

$du = \cos(x)\,dx$

$$\int 4\,u^{1/2}\,du = 4\cdot\frac{2}{3}u^{3/2} + C = \boxed{\frac{8}{3}(\sin(x)+1)^{3/2} + C}$$

(d) $\int x^3\sqrt{x^2+1}\, dx$

$u = x^2+1 \longleftrightarrow x^2 = u-1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$\int x^3\sqrt{u}\,\frac{du}{2x}$$

$$\int x^2\sqrt{u}\,du$$

$$\frac{1}{2}\int x^2\cdot 2x\sqrt{x^2+1}\,dx = \frac{1}{2}\int (u-1)\,u^{1/2}\,du$$

$$= \frac{1}{2}\int\left(u^{3/2} - u^{1/2}\right)du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C = \boxed{\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C}$$

8. Evaluate

(a) $\int \tan x \cdot \sec^2(x)\, dx$

$u = \tan(x)$

$du = \sec^2(x)\, dx$

$\int u\, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2(x) + C}$

$u = \sec(x)$

$du = \sec(x)\tan(x)\, dx$

Also works

$\int u\, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sec^2(x) + C}$

$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$

$\tan^2(x) + 1 = \sec^2(x)$

(b) $\int \tan x \cdot \sec^2(x)\, dx = \int \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos^2(x)}\, dx = -\int \frac{(-)\sin(x)}{\cos^3(x)}\, dx$

$u = \cos(x)$

$du = -\sin(x)\, dx$

$-\int \frac{1}{u^3}\, du = -\int u^{-3}\, du = -\left(-\frac{1}{2}u^{-2}\right) + C = \boxed{\frac{1}{2}\sec^2(x) + C}$

9. Evaluate

(a) $\frac{1}{8}\int 8 e^{8x}\, dx$

$u = 8x$

$du = 8\, dx$

$\frac{1}{8}\int e^u\, du = \frac{1}{8}e^u + C = \boxed{\frac{1}{8}e^{8x} + C}$

(b) $\int_{1/8}^{2} e^{8x}\, dx = \frac{1}{8}e^{8x}\Big|_{1/8}^{2} = \frac{1}{8}e^{8(2)} - \frac{1}{8}e^{8(\frac{1}{8})}$

$= \boxed{\frac{1}{8}e^{16} - \frac{1}{8}e = \frac{1}{8}\left(e^{16} - e\right)}$

*Typo

(c) $\frac{1}{8}\int_{1/8}^{2} 8 e^{8x}\, dx$

$u = 8x$

$du = 8\, dx$

Bounds: $u(2) = 8(2) = 16$

$u(1/8) = 8(1/8) = 1$

$\frac{1}{8}\int_1^{16} e^u\, du = \frac{1}{8}e^u\Big|_1^{16} = \boxed{\frac{1}{8}\left(e^{16} - e\right)}$

10. Evaluate

(a) $\frac{1}{6}\int_0^1 6t^2(2t^3+3)^{20}\,dt$

$u = 2t^3 + 3$

$du = 6t^2\,dt$

Bounds: $u(1) = 2(1)^3 + 3 = 5$

$u(0) = 2(0)^3 + 3 = 3$

$$\frac{1}{6}\int_3^5 u^{20}\,du = \frac{1}{6}\cdot\frac{1}{21}u^{21}\Big|_3^5$$

$$\boxed{\frac{1}{126}\left(5^{21} - 3^{21}\right)}$$

(b) $\frac{1}{2}\int_1^2 x(2x+5)^{10}\,2\,dx$

$u = 2x+5 \leftrightarrow 2x = u-5$

$du = 2\,dx \qquad x = \frac{1}{2}(u-5)$

Bounds: $u(2) = 2(2)+5 = 9$

$u(1) = 2(1)+5 = 7$

$$= \frac{1}{2}\int_7^9 \frac{1}{2}(u-5)u^{10}\,du$$

$$= \frac{1}{4}\int_7^9 (u^{11} - 5u^{10})\,du = \frac{1}{4}\left[\frac{1}{12}u^{12} - \frac{5}{11}u^{11}\right]_7^9$$

$$\boxed{\frac{1}{4}\left[\frac{1}{12}(9^{12}) - \frac{5}{11}(9^{11})\right] - \frac{1}{4}\left[\frac{1}{12}(7^{12}) - \frac{5}{11}(7^{11})\right]}$$

(c) $\int_1^4 \frac{1}{(x+1)\sqrt{x}}\,dx$ $= 2\int_1^4 \frac{1}{x+1} \cdot \frac{1}{2\sqrt{x}}\,dx$

$u = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\,dx$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{x}}\,dx$

$= 2\int_1^4 \frac{1}{(x^{1/2})^2+1}\,\frac{1}{2\sqrt{x}}\,dx$

Bounds:

$u(4) = \sqrt{4} = 2$

$u(1) = \sqrt{1} = 1$

$= 2\int_1^2 \frac{1}{u^2+1}\,du$

$= 2\arctan(u)\Big|_1^2 = \boxed{2\arctan(2) - 2\arctan(1)}$

(d) $\int_0^1 \frac{1}{(1+\sqrt{x})^4}\,dx$

$u = 1 + x^{1/2} \longleftrightarrow \sqrt{x} = u-1$

$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$

$\int_0^1 \frac{1}{(1+\sqrt{x})^4}\,\frac{2\sqrt{x}}{2\sqrt{x}}\,dx$

Bounds: $u(1) = 1+\sqrt{1} = 2$

$u(0) = 1+\sqrt{0} = 1$

$= \int_1^2 \frac{2(u-1)}{u^4}\,du = 2\int_1^2 \left(\frac{u}{u^4} - \frac{1}{u^4}\right)du = 2\int_1^2 (u^{-3} - u^{-4})\,du$

$2\left[\left(-\frac{1}{2}u^{-2}\right) - \left(\frac{-1}{3}u^{-3}\right)\right]_1^2$

$\int \frac{\arctan(4x)}{1+16x^2}\,dx$

$= \int \arctan(4x)\cdot\frac{1}{1+16x^2}\,dx$