Math 308: Week-in-Review 12

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Review for the Final Exam - Part 1

1. (Chapter 2) Find the solution to the initial value problem

(a) $2\sqrt{x}\dfrac{dy}{dx} = \cos^2 y, \qquad y(4) = \dfrac{\pi}{4}$

* nonlinear, first order, separable *

$$\int \frac{1}{\cos^2 y}\,dy = \frac{1}{2}\int \frac{1}{\sqrt{x}}\,dx$$

$$\int \sec^2 y\,dy = \frac{1}{2}\int x^{-1/2}\,dx$$

$$\tan y = \frac{1}{2}\cdot 2 \cdot x^{1/2} + C = \sqrt{x} + C$$

Check equilibrium solutions

$2\sqrt{x}\,y' = \cos^2 y = 0 \Rightarrow \cos^2 y = 0$

$y(x) = \pm(2n+1)\frac{\pi}{2}$ (odd multiples of $\frac{\pi}{2}$) for $n = 0, 1, 2, 3, \ldots$

None of these are $\frac{\pi}{4}$

* Find C * $\quad y(4) = \frac{\pi}{4} \Rightarrow x = 4,\ y = \frac{\pi}{4}$

$$\tan\frac{\pi}{4} = \sqrt{4} + C \Rightarrow 1 = 2 + C \Rightarrow C = -1$$

$$\tan y = \sqrt{x} - 1$$

$$y = \tan^{-1}(\sqrt{x} - 1) = \arctan(\sqrt{x} - 1)$$

* Check * $\quad y(4) = \tan^{-1}(\sqrt{4} - 1) = \tan^{-1}(2-1) = \tan^{-1}(1)$

$y(4) = \frac{\pi}{2}$

(b) $\underbrace{\dfrac{dy}{dt} + \dfrac{2y}{t}}\;= \dfrac{\cos t}{t^2}, \qquad y(1) = \dfrac{1}{2}, \qquad t > 0.$

first order linear, nonhomogeneous

$y'(t) + p(t)y = g(t) \rightarrow$ standard form

integrating factor

$$\boxed{\begin{aligned} \mu(t) &= e^{\int p(t)dt} \\ &= e^{\int 2/t\, dt} = e^{2\int 1/t\, dt} \\ &= e^{2\ln t} = e^{\ln t^2} = t^2 \end{aligned}}$$

$\mu(t) = t^2$

$$y(t) = \frac{1}{\mu(t)}\int \mu(t) g(t)\,dt + \frac{C}{\mu(t)}$$

$$= \frac{1}{t^2}\int t^2 \cdot \frac{\cos t}{t^2}\,dt + \frac{C}{t^2}$$

$$= \frac{1}{t^2}\int \cos t\,dt + \frac{C}{t^2}$$

$$y(t) = \frac{1}{t^2}\sin t + \frac{C}{t^2} \Rightarrow y(1) = \sin(1) + C = 1/2 \Rightarrow C = 1/2 - \sin(1)$$

$$y(t) = \frac{\sin t}{t^2} + \frac{\left(\frac{1}{2} - \sin(1)\right)}{t^2}$$

(c) $y' + 2y = e^{-2t}$, $\qquad y(0) = 1$.

* first order, linear, nonhomogeneous *

$$p(t) = e^{\int 2\,dt} = e^{2t}$$

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)\, g(t)\, dt + \frac{C}{\mu(t)}$$

$$= e^{-2t} \int e^{2t} \cdot e^{-2t}\, dt + Ce^{-2t}$$

$$= e^{-2t} \cdot t + Ce^{-2t}$$

$$y(0) = C = 1$$

$$\boxed{y(t) = e^{-2t}(t+1)}$$

* check * $\quad y'(t) = e^{-2t} - 2e^{-2t}(t+1)$

$$= e^{-2t} - 2te^{-2t} - 2e^{-2t}$$

$$= -e^{-2t} - 2te^{-2t}$$

$$y' + 2y = (-e^{-2t} - \cancel{2te^{-2t}}) + \cancel{2te^{-2t}} + 2e^{-2t}$$

$$= e^{-2t} \checkmark$$

2. (Chapter 1, 2) A tank contains 100 gal of brine in which 50 lbs of salt are dissolved. Brine containing 2 lbs of salt per galon flows into the tank at a rate of 6 gal per min. The mixture, which is kept uniform by stirring, flows out of the tank at the rate of 4 gal per min. Volume $= 100+2t$ gal

(a) Find the amount of salt in the tank at the end of t minutes.

$$Q'(t) = \text{rate in} - \text{rate out} = \left(2\,\frac{\text{lbs}}{\text{gal}}\right)\left(6\,\frac{\text{gal}}{\text{min}}\right) - \frac{Q(t)}{100+2t}\,\frac{\text{lbs}}{\text{gal}} * 4\,\frac{\text{gal}}{\text{min}}$$

$$Q'(t) = 12 - \frac{2Q(t)}{50+t}, \quad Q(0) = 50 \text{ lbs}$$

$$Q'(t) + \frac{2Q(t)}{50+t} = 12, \quad Q(0) = 50 \quad \Rightarrow \mu(t) = e^{2\int \frac{1}{50+t}dt} = e^{2\ln(50+t)} = (50+t)^2$$

$$Q(t) = (50+t)^{-2}\int 12(50+t)^2\,dt + C(50+t)^{-2} = 4(50+t) + C(50+t)^{-2}$$

$$Q(0) = 200 + \frac{C}{50^2} = 50 \Rightarrow \frac{C}{50^2} = -150 = (-3)(50) \Rightarrow C = (-3)(50^3)$$

$$\boxed{Q(t) = 4(50+t) - \frac{3(50^3)}{(50+t)^2}}$$

(b) After 50 mins, how much salt will be in the tank, and what will be the volume of brine?

$$Q(50) = 4(50+50) - \frac{3(50^3)}{4 \cdot 50^2} = 400 - \frac{3}{4}(50)$$

$$= 400 - 37.5$$

$$= 362.5 \text{ lbs}$$

3. (Chapter 2) Given the differential equation

$$\frac{dy}{dt} = y^3 - 2y^2 + y = y(y^2-2y+1) = y(y-1)^2$$

(a) Find the equilibrium solutions

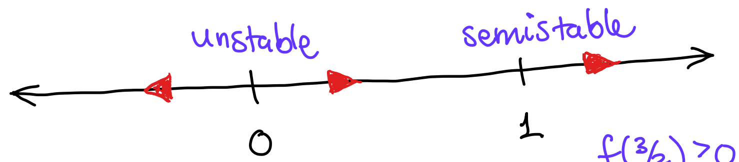
(b) Graph the phase line. Classify each equilibrium solution as either stable, unstable, or semi-stable.

(c) Sketch the graph of some solutions.

(d) If $y(t)$ is the solution of the equation satisfying the initial condition $y(0) = y_0$, where $-\infty < y_0 < \infty$, find the limit of $y(t)$ when t increases.

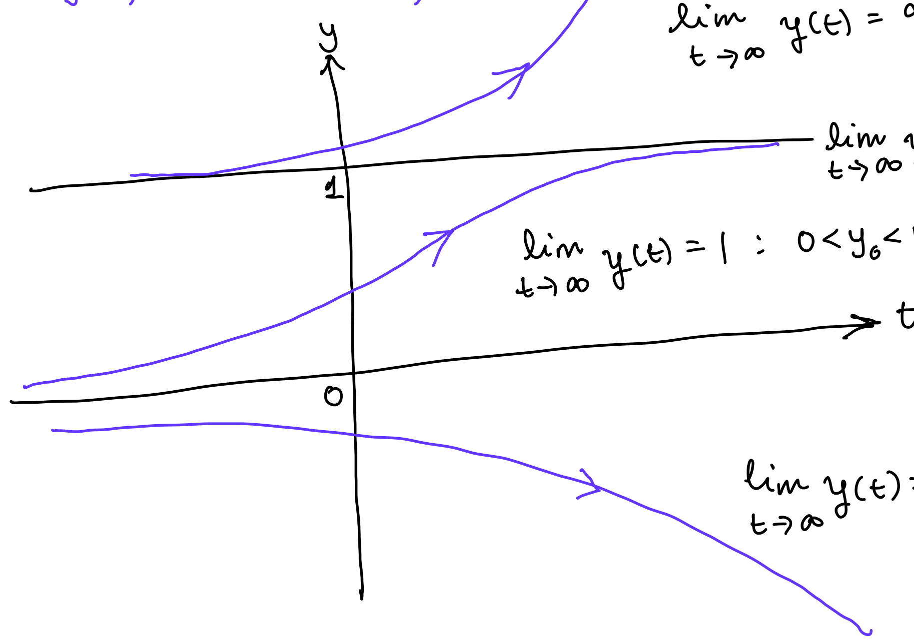
(a) $f(y) = y(y-1)^2 = 0 \Rightarrow y = 0,\ y = 1$

(b)



$f(-1) < 0$ $\quad f(\tfrac{1}{2}) > 0$ $\quad f(3/2) > 0$

(c)



$\lim_{t\to\infty} y(t) = \infty$: $y_0 > 1$

$\lim_{t\to\infty} y(t) = 1$: $y_0 = 1$

$\lim_{t\to\infty} y(t) = 1$: $0 < y_0 < 1$

$\lim_{t\to\infty} y(t) = 0$: $y_0 = 0$

$\lim_{t\to\infty} y(t) = -\infty$: $y_0 < 0$

4. (Chapter 2) Find an integrating factor for the equation

$$\underbrace{(3xy + y^2)}_{M}dx + \underbrace{(x^2 + xy)}_{N}dy = 0$$

and then solve the equation.

$M_y = 3x+2y \neq N_x = 2x+y$

$\dfrac{M_y - N_x}{N}$ (x only) OR $\dfrac{N_x - M_y}{M}$ (y-only)

$$\frac{(3x+2y)-(2x+y)}{x^2+xy} = \frac{\cancel{x}+y}{x(\cancel{x+}y)} = \frac{1}{x} \Rightarrow \mu(x) = e^{\int \frac{M_y - N_x}{N}dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$\mu(x) = x$

$$x(3xy+y^2)dx + x(x^2+xy)dy = 0$$

$$\Rightarrow \underbrace{(3x^2y + xy^2)}_{M}dx + \underbrace{(x^3+x^2y)}_{N}dy = 0$$

$\begin{cases} M_y = 3x^2+2xy \\ N_x = 3x^2+2xy \end{cases}$ exact

$$F_x = M = 3x^2y + xy^2 \Rightarrow F(x,y) = \int M(x,y)dx = \int (3x^2y+xy^2)dx = x^3y + \tfrac{1}{2}x^2y^2 + h(y)$$

$$F_y = x^3 + x^2y + h'(y) = N \Rightarrow h'(y) = 0 \Rightarrow h(y) = 0$$

$F(x,y) = x^3y + \frac{1}{2}x^2y^2 = C$ (implicit form)

5. (Chapter 3) Find the general solution of the equation $y'' + 2y' + y = 4e^{-t}$.

$y(t) = y_c(t) + y_p(t)$, $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$, $y_p(t) = At^2 e^{-t}$

$y_p' = 2Ate^{-t} - At^2 e^{-t}$, $y_p'' = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2 e^{-t}$

$y_p'' + 2y_p' + y_p = (2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2 e^{-t}}) + 2(\cancel{2Ate^{-t}} - \cancel{At^2 e^{-t}}) + \cancel{At^2 e^{-t}}$

$= 2Ae^{-t} = 4e^{-t} \Rightarrow A = 2$ $\qquad y_p(t) = 2t^2 e^{-t}$

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + 2t^2 e^{-t}$

6. (Chapter 3) Find the form of a particular solution for each of the following nonhomogeneous equations. Do **not** solve the equation.

(a) $y'' + 2y' + 2y = e^{-t}\sin t + e^{-t}\cos 2t$ $\quad$ homogeneous $\quad \lambda = \frac{-2 \pm \sqrt{4 - 4\cdot 2}}{2} = -1 \pm i$

(b) $y'' - 2y' + y = te^t + t^2 e^{-t} + e^t \cos t + t^2$ $\quad$ homogeneous $\quad \lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$ (repeated)

(a) $y_c(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$

$y_p(t) = t\left[Ae^{-t}\cos(t) + Be^{-t}\sin(t)\right] + \left[Ce^{-t}\cos(2t) + De^{-t}\sin(2t)\right]$

(b) $y_c(t) = c_1 e^{t} + c_2 t e^{t}$

$y_p(t) = t^2(A + Bt)e^{t} + (Ct^2 + Dt + E)e^{-t}$

$+ (Fe^{-t}\cos t + Ge^{-t}\sin t) + (Ht^2 + It + J)$

7. (Chapter 3) Find the general solution of the equation

$$y'' + 6y' + 9y = \frac{e^{-3x}}{1+2x}$$

not a quasi-polynomial

* variation of parameters *

$y(x) = y_c(x) + y_p(x)$

$y_c(x) = c_1 e^{-3x} + c_2 x e^{-3x} \Rightarrow y_1(x) = e^{-3x}, \; y_2(x) = x e^{-3x}$

$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$ where $\begin{cases} u_1(x) = \int -\dfrac{y_2(x) r(x)}{W(x)} dx \\ u_2(x) = \int \dfrac{y_1(x) r(x)}{W(x)} dx \end{cases}$

$$W(x) = \begin{vmatrix} e^{-3x} & x e^{-3x} \\ -3e^{-3x} & e^{-3x} - 3x e^{-3x} \end{vmatrix} = e^{-6x} - 3\cancel{x} e^{-6x} + 3\cancel{x} e^{-6x} = e^{-6x}$$

$$u_1(x) = \int \frac{-x e^{-3x} \cdot \dfrac{e^{-3x}}{1+2x}}{e^{6x}} dx$$

$$= -\int \frac{x}{1+2x} dx$$

$u = 1+2x \Rightarrow du = 2dx$

$x = \dfrac{(u-1)}{2}$

$$= -\int \frac{\frac{(u-1)}{2} \cdot \frac{1}{2} du}{u} = -\frac{1}{4} \int \frac{u-1}{u} du$$

$$= -\frac{1}{4}\left(u - \ln|u|\right)$$

$$= -\frac{1}{4}(1+2x) + \frac{1}{4} \ln|1+2x|$$

$$u_2(x) = \int \frac{e^{-3x} \cdot \dfrac{e^{-3x}}{1+2x}}{e^{-6x}} dx = \int \frac{1}{1+2x} dx = \frac{1}{2} \ln|1+2x|$$

$$y_p(x) = -\frac{1}{4} e^{-3x}(1+2x) + \frac{1}{4} e^{-3x} \ln|1+2x| + \frac{1}{2} x e^{-3x} \ln|1+2x|$$

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} - \frac{1}{4} e^{-3x}(1+2x) + \frac{1}{4} e^{-3x} \ln|1+2x| + \frac{1}{2} x e^{-3x} \ln|1+2x|$$