Math 152 - Final Exam Review

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List of Topics for the Final Exam

* **Methods of Integration:**
  1. List of anti derivatives
  2. u-substitution
  3. Integration by parts $uv - \int v\,du$ ILATE
  4. Trigonometric Integrals → trig identities $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$
  5. Trigonometric substitution $a\sin\theta,\ a\tan\theta,\ a\sec\theta$
  6. Partial Fraction Decomposition
  7. Improper Integrals

* **Applications of Integration:**
  1. Areas between curves T–B → dx, R–L → dy
  2. Volume of a solid by rotation: method of disks, washers or cylindrical shells $2\pi rh$ πR^2 or $\pi(R^2-r^2)$
  3. Volume of a solid by slices
  4. Work: tanks, springs, ropes

	Disks/washers	Shells
→	$\int dx$	$\int dy$
↑	$\int dy$	$\int dx$

Sequences: if $\lim_{n\to\infty} a_n = L$

1. Convergence and divergence
2. Increasing or decreasing sequences
3. Boundedness of a sequence
4. Alternating sequences
5. Recursive sequences

Series → sum of a sequence $\sum_{n=1}^{\infty} a_n$

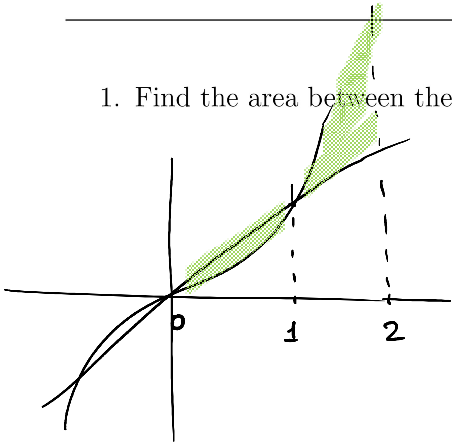
Converge if $\sum a_n = S = L$

1. The Partial sum s_n of a series
2. The telescoping series
3. The Geometric series $S = \sum ar^n = \frac{a_1}{1-r}$
4. The test for Divergence (TOD)
5. The Integral test and remainder estimate
6. The Comparison and Limit Comparison tests
7. The Alternating series test and error estimate $(-1)^n$ or $(-1)^{n\pm1}$
8. The Ratio test and absolute convergence exp or factorials.
9. The Taylor and Maclaurin series
10. Taylor Polynomials

Parametric and Polar curves

1. Parametric curves
2. Arc Length and Surface area
3. Polar coordinates
4. Areas in polar coordinates

1. Find the area between the curves $y = x^3$ and $y = x$ for $0 \le x \le 2.$



Intersection points

$x^3 = x$ or $x^3 - x = 0$.

$x(x^2-1) = 0$.

(T−B) (T−B)

0 1 2

$x = 0, x = \pm 1$

$$A = \int_0^1 (x - x^3)\,dx + \int_1^2 (x^3 - x)\,dx$$

$$= \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^1 + \frac{x^4}{4} - \frac{x^2}{2}\Big|_1^2$$

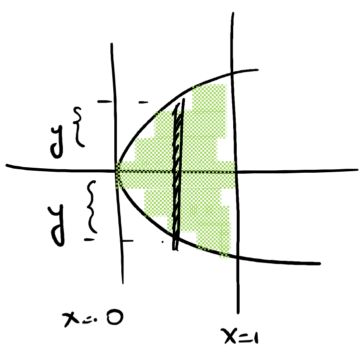
$$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left[(4-2) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + 2 + \frac{1}{4} = \frac{5}{2}$$ Ans.

2. The base of a solid S is given by area enclosed by the curves $x = y^2$ and $x \le 1.$ Cross sections perpendicular to the x-axis are squares. Find the volume of the solid S.

$\int dx$

Squares



$\ell = 2y$

$A = \ell^2 = 4y^2$

$A = 4x$

$$V = \int_0^1 A\,dx = \int_0^1 4x\,dx$$

$$= \frac{4x^2}{2}\Big|_0^1 = 2$$ Ans.

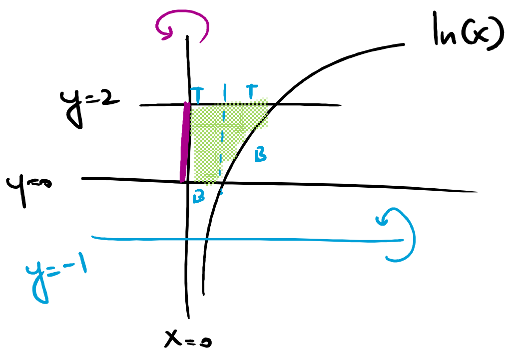
Graphs to refresh: $f(x) = \ln(x)$

$f(x) = \arctan(x)$ or $\tan(x)$

$y = \ln(x) \longleftrightarrow x = e^y$

3. Find the volume of a solid formed by rotating the region bounded by the curves $x = 0$, $y = \ln(x)$, $y = 0$, $y = 2$ about

   (a) the y-axis.



✓ Disks $\to \int dy$ Shells $\to \int dx$

$$V = \pi \int_0^2 (e^y)^2\,dy = \pi \int_0^2 e^{2y}\,dy$$

$$= \pi \cdot \frac{e^{2y}}{2} \Big|_0^2$$

$$= \frac{\pi}{2}\left(e^4 - 1\right)$$

   (b) the line $y = -1$.

Washers $\to \int dx$ $\Downarrow$ 2 integrals

✓ Shells $\to \int dy$ $\Downarrow$ 1 integral

$r = T - B$ $h = R - L$

$$V = 2\pi \int_0^2 (y+1)(e^y - 0)\,dy$$

$$= 2\pi \int_0^2 (y+1)e^y\,dy = 2\pi \int_0^2 ye^y\,dy + 2\pi \int_0^2 e^y\,dy$$

$$= 2\pi\left[ye^y - e^y \Big|_0^2\right] + 2\pi(e^2 - 1)$$

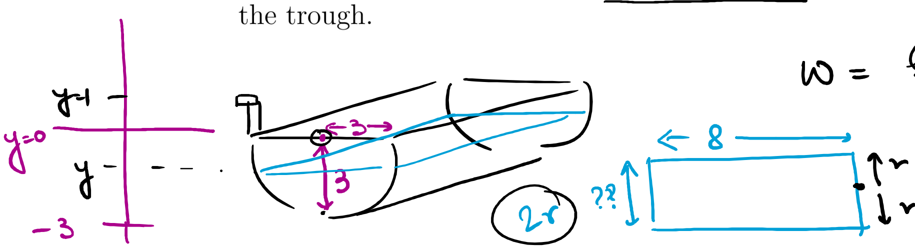
$$= 2\pi\left[2e^2 - e^2 - (0 - 1)\right] + 2\pi(e^2 - 1)$$

$$= 4\pi e^2 - \cancel{2\pi e^2} + \cancel{2\pi} + \cancel{2\pi e^2} - \cancel{2\pi}$$

$$= 4\pi e^2$$ Ans.

Tank.

4. Consider a trough in the shape of a halfcylinder of radius 3 feet and length 8 feet (diameter at the top). It is full of water to a depth of 3 feet. Find an integral that gives the work necessary to pump all of the water to a point 1 foot above the top of the trough.



$$W = \rho g \int_{-3}^{0} A_s\,(T-y)\,dy.$$

$T = 1.$

$$W = \rho g \int_{-3}^{0} A_s\,(1-y)\,dy$$

$$A_s = (8)(\text{width}) = (8)(2)(\sqrt{9-y^2})$$

diameter $= 2r$. from spherical tank.

$$r = \sqrt{R^2 - y^2} = \sqrt{(3)^2 - y^2}$$
$$= \sqrt{9-y^2}$$

Ans.

$$W = \rho g \int_{-3}^{0} 16\sqrt{9-y^2}\,(1-y)\,dy$$

OR $$W = \rho g \int_{0}^{3} 16\sqrt{9-y^2}\,(1+y)\,dy$$

5. A spring has a natural length of 3 meters. A force of 10 N is required to keep the spring stretched an additional 50 cm. Find the amount of work required to stretch the spring from its natural length to a length of 5m. $\longrightarrow$ from 3 to 5 m.

$$f = kx$$
$$10 = k\left(\frac{50}{100}\right) = 0.5k.$$
$$k = \frac{10}{0.5} = 20$$

$$W = \frac{1}{2}kx^2 \Big|_{3-3=0}^{5-3=2}$$
$$= \frac{1}{2}(20)x^2 \Big|_0^2$$
$$= 10[4-0]$$
$$= 40 \text{ Joules}.$$

6. An 800-lb steel beam hangs from a 50-foot cable which weighs 6 pounds/foot. Find the work done in winding up 20 feet of the cable.

δ

50 ft of cable weighs $50 \times 6 = 300$ lbs.

Initial wt $=$ $\underset{\text{beam}}{800} + \underset{\text{cable}}{300} = 1100$ lbs.

$$W = \int_0^{20} (1100 - \delta y)\,dy = \int_0^{20} (1100 - 6y)\,dy$$

$$= 1100y\Big|_0^{20} - \frac{\cancel{6}^{3} y^2}{\cancel{2}}\Big|_0^{20}$$

$$= 1100(20-0) - 3(400-0)$$

$$= 22000 - 1200 = 20{,}800 \text{ ft-lb}.$$

7. Evaluate $\int_0^1 x^2 e^{-2x}\,dx$.

use $t = -2x$.
$dt = -2dx$
$\rightarrow$ try by yourself.

Tabular

$\downarrow$ u		$v \rightarrow \int$
x^2	+	e^{-2x}
$2x$	−	$e^{-2x}/(-2)$
2	+	$e^{-2x}/(-2)^2$
0		$e^{-2x}/(-2)^3$

$$\frac{x^2 e^{-2x}}{(-2)} - \frac{\cancel{2}x \cdot e^{-2x}}{\cancel{4}_2} + \frac{\cancel{2}e^{-2x}}{(-\cancel{8}_4)}\Bigg|_0^1$$

$$= \frac{e^{-2}}{-2} - \frac{e^{-2}}{2} - \frac{e^{-2}}{4} - \left(0 - 0 - \frac{1}{4}\right)$$

$$= e^{-2}\left(-\frac{5}{4}\right) + \frac{1}{4}$$

$$= \frac{1}{4}\left(-5e^{-2} + 1\right)$$

8. Compute the following improper integral or show that it diverges: $\int_4^{\infty} \frac{x+7}{x^2-x-6}\,dx$

$$\frac{x+7}{x^2-x-6} = \frac{x+7}{(x+2)(x-3)} \rightarrow \text{Partial Fractions}$$

$$\left[\frac{x+7}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)}\right](x+2)(x-3)$$

$$x+7 = A(x-3) + B(x+2)$$

@ $x=3$, $\quad 10 = B(5)$ or $B = 2$

@ $x=-2$, $\quad 5 = A(-5)$ or $A=-1$

Int: $\displaystyle\int_4^{\infty} \frac{-1}{x+2}\,dx + \int_4^{8} \frac{2}{x-3}\,dx$

$$= -\ln(x+2)\Big|_4^{\infty}$$

$$= -\ln(\infty) + \ln(6)$$

$$= -\infty$$

$\therefore$ Integral diverges.

9. Evaluate $\int_{x=0}^{x=\pi/2} \sin^2(x)\cos^3(x)\, dx$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$= 1 - u^2$

$u = \sin(x)$

$du = \cos(x)dx$

$= \int u^2 (\cos^2 x)\, du$

$= \int u^2(1-u^2)\, du = \int u^2 du - \int u^4 du$

$= \frac{u^3}{3} - \frac{u^5}{5}$

$= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) \Big|_0^{\pi/2}$

$= \frac{1}{3} - \frac{1}{5}$

$= \frac{2}{15}$ Ans.

10. Evaluate $\int_1^e \frac{\sqrt{\ln(x)}}{x}\, dx$

$= \int_1^e \frac{1}{x} \cdot \sqrt{\ln(x)}\, dx$

$u = \ln(x)$

$du = \frac{1}{x} dx$

$= \int \sqrt{u}\, du = \int u^{1/2} du$

$= \frac{u^{3/2}}{3/2}$

$= \frac{2}{3} \ln(x)^{3/2} \Big|_1^e = \frac{2}{3}\left[1^{3/2} - 0\right] = \frac{2}{3}$ Ans.

11. Evaluate $\displaystyle\int \frac{x^3}{\sqrt{4x^2-9}}\,dx$

converges if $\lim_{n\to\infty} a_n$ is finite

12. Find the limit of the following sequences:

(a) $a_n = \arctan\left(\dfrac{n}{n+1}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan\left(\frac{n}{n+1}\right) = \arctan\left(\cancel{\lim_{n\to\infty} \frac{n}{n+1}}^{\,1}\right)$$

$$= \arctan(1)$$

$$= \pi/4 \rightarrow \text{finite} \quad \therefore \text{ sequence converges.}$$

(b) $a_n = \dfrac{(-1)^{n+1}}{2n+1}$ $\longrightarrow$ alternating sequence

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{2n+1} \right| = \lim_{n\to\infty} \frac{1}{2n+1} = 0.$$

sequence converges to zero.

(c) $a_1 = 1,\ a_{n+1} = \sqrt{3 + a_n}$ $\longrightarrow$ recursive sequence.

$$\lim_{n\to\infty} a_n = L$$

$$\lim_{n\to\infty} a_{n+1} = L$$

$$\lim_{n\to\infty} \left(a_{n+1} = \sqrt{3 + a_n} \right)$$

$a_1 = 1$
$a_2 = \sqrt{3+1} = 2$
$a_3 = \sqrt{3+2} = \sqrt{5}$
$a_4 = \sqrt{3+\sqrt{5}}$

increasing

$L = 0.5 + \dfrac{\sqrt{13}}{2}$

$$L = \sqrt{3 + L}$$

$$L^2 = 3 + L$$

$$L^2 - L - 3 = 0.$$

$$L = \frac{+1 \pm \sqrt{1 - 4(1)(-3)}}{2} = \frac{1 \pm \sqrt{13}}{2} = 0.5 \pm \tfrac{1}{2}\sqrt{13}$$

~ 2

13. Find the sum of the series $\sum_{n=1}^{\infty} \frac{2^n + (-4)^n}{6^n}$.

$$= \overbrace{\sum_{n=1}^{\infty} \left(\frac{2}{6}\right)^n}^{S_1} + \overbrace{\sum_{n=1}^{\infty} \left(\frac{-4}{6}\right)^n}^{S_2}$$

$$\underbrace{r = \frac{1}{3} < 1}_{\text{conv}} \qquad \underbrace{|r| = \frac{2}{3} < 1}_{\text{conv.}}$$

$\therefore$ Sum

$$S_1 = \frac{\left(\frac{2}{6}\right)^1}{1 - \left(\frac{1}{3}\right)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$S_2 = \frac{\left(-\frac{4}{6}\right)^1}{1 - \left(-\frac{4}{6}\right)} = \frac{-2/3}{1 + 2/3} = \frac{-2/3}{5/3} = -\frac{2}{5}$$

$$S = S_1 + S_2 = \frac{1}{2} - \frac{2}{5} = \frac{5-4}{10} = \frac{1}{10}$$

Ans.

14. Find a_5 and the sum of the series s if the partial sum of the series is $s_n = \frac{3n+2}{1-2n}$

Given S_n, $S = \lim_{n\to\infty} S_n$

$$= \lim_{n\to\infty} \left(\frac{3n+2}{1-2n}\right) = -\frac{3}{2}$$

finite value
$\therefore$ series converges

$$a_5 = S_5 - S_4$$

$$= \left(\frac{15+2}{1-10}\right) - \left(\frac{12+2}{1-8}\right)$$

$$= \frac{17}{-9} + \frac{14}{7}$$

$n=3$

15. Use the third partial sum to approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$. What is the maximum error?

uses Integral Test

$$S = S_3 + R_3$$

$$R_3 = \int_3^{\infty} \frac{1}{x^4}\,dx \qquad \sum \frac{1}{n^4} = \int \frac{1}{x^4}\,dx$$

$$= \int_3^{\infty} x^{-4}\,dx$$

$$= \left.\frac{x^{-3}}{-3}\right|_3^{\infty} = -\frac{1}{3}\left(\cancel{\frac{1}{\infty^3}}^{\,0} - \frac{1}{3^3}\right) = \frac{1}{3^4}$$

max error

$n=3$

16. Use the third partial sum to approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$. What is the maximum error?

use Alternating Series error estimate

$$S = S_3 + R_3$$

$$S_3 = a_1 + a_2 + a_3$$

$$|R_3| \leq a_4$$

$$R_3 = \frac{1}{4^4}$$ $\rightarrow$ max error.

17. Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{\sqrt{n}}$ absolutely convergent, ~~partially~~ conditionally convergent or divergent?

$\sum |a_n| = \sum \frac{e^{1/n}}{\sqrt{n}}$ diverges by LCT to $\sum \frac{1}{\sqrt{n}}$

But $\sum a_n$ converges by AST.

$\therefore$ series is conditionally convergent

18. For which of the following series does the Ratio test fail?

exponents or factorials.

(a) $\sum_{n=1}^{\infty} \frac{2n+5}{3n^3-7}$ $\rightarrow$ not a RT problem.

alternating $\rightarrow$ not a candidate for RT.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

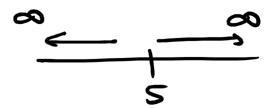
RT: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

✓ (c) $\sum_{n=1}^{\infty} \frac{1}{(-2)^n(n^2+1)}$

✓ (d) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

19. Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-5)^n}{n!}$.

$a = 5$



$$\text{RT: } \lim_{n\to\infty} \left| \frac{\cancel{3^{n+1}}^{\,3}\ \cancel{(x-5)^{n+1}}^{\,(x-5)}}{\underbrace{\cancel{(n+1)!}}_{(n+1)}} \cdot \frac{\cancel{n!}}{\cancel{3^n}\ \cancel{(x-5)^n}} \right|$$

$$|3(x-5)| \lim_{n\to\infty} \left(\cancel{\frac{1}{n+1}}^{\,0} \right) = 0 \rightarrow \text{series will always converge.}$$

$\therefore R = \infty$

IC: $(-\infty, \infty)$

20. Find the radius of convergence of the Taylor series for the function $f(x) = x\ln(1+x^2)$, centered about zero.

Maclaurin

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2}}{(n+1)}$$

$$f(x) = x\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+3}}{(n+1)}$$

$$\text{RT: } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+5}}{(n+2)} \cdot \frac{(n+1)}{x^{2n+3}} \right| = |x^2|$$

for convergence $|x^2| < 1$

or $|x| < 1$

$\therefore$ Radius $= 1$

IC: $(-1, 1)$

21. Find the Maclaurin series for the function $f(x) = \dfrac{x^2}{(1-3x)^2}$.

$f(x) = x^2 \cdot g(x)$

$g(x) = \dfrac{1}{(1-3x)^2}$

$\int g(x)\,dx = \int \dfrac{1}{(1-3x)^2}\,dx$

$u = 1-3x$

$du = -3\,dx$.

$= \int \dfrac{1}{u^2} \cdot \dfrac{du}{(-3)}$

$= -\frac{1}{3} \cdot \left(-\frac{1}{u}\right)$

$= \frac{1}{3}\left(\frac{1}{1-3x}\right)$

$\int g(x)\,dx = \frac{1}{3} \cdot \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} \frac{3^n}{3} \cdot x^n$

$= \sum_{n=0}^{\infty} 3^{n-1} x^n$

$g(x) = \frac{d}{dx} \sum_{n=0}^{\infty} 3^{n-1} x^n = \sum_{n=1}^{\infty} 3^{n-1} \cdot n x^{n-1}$

$f(x) = x^2 \cdot g(x) = \sum_{n=0}^{\infty} 3^n (n+1) \cdot x^n$

$= \sum_{n=0}^{\infty} 3^n (n+1) x^{n+2}$

22. Find the Taylor series for $f(x) = \dfrac{1}{x^2}$ about $a = 5$.

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(5)}{n!} (x-5)^n$.

$f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = (-2)x^{-3}$

$f''(x) = (-2)(-3)x^{-4}$

$f'''(x) = (-2)(-3)(-4)x^{-5}$

$f^n(x) = (-1)^n \cdot (n+1)!\, x^{-(n+2)}$

$f^n(5) = \dfrac{(-1)^n (n+1)!}{5^{n+2}}$

$\therefore f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n+1)!}{5^{n+2}} \cdot \frac{(x-5)^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (n+1)}{5^{n+2}} \cdot (x-5)^n$

23. Find the arclength of the parametric curve given by $x = (\sqrt{2}/3)t^{3/2}$, $y = t + 27$ from $t = 0$ to $t = 6$.

$$x = \frac{\sqrt{2}}{3} t^{3/2} \qquad y = t+27$$

$$x' = \frac{\sqrt{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{2} t^{1/2} \qquad y' = 1$$

$$(x')^2 + (y')^2 = \frac{\cancel{2}}{\cancel{4}_2} t + 1 = \frac{1}{2}t + 1$$

$$\therefore L = \int_0^6 \sqrt{\tfrac{1}{2}t+1}\, dt$$

$$u = \tfrac{1}{2}t + 1$$
$$du = \tfrac{1}{2}dt$$

$$= \frac{\left(\frac{1}{2}t+1\right)^{3/2}}{3/2} \cdot 2 \Bigg|_{t=0}^{t=6}$$

$$= \frac{4}{3}\left[4^{3/2} - 1\right] = \frac{4}{3}(8-1) = \frac{28}{3}$$

Ans.

24. Find the surface area of the object obatined by rotating the curve $\underbrace{y = e^{x/2},\ 0 \le x \le 2}$ about the x-axis.

$$x = t, \quad y = e^{t/2}, \quad 0 \le t \le 2$$

$$r = y = e^{x/2}$$

$$x' = 1 \qquad y' = \frac{1}{2}e^{t/2}$$

$$(x')^2 + (y')^2 = 1 + \frac{1}{4}e^t$$

$$S = 2\pi \int_0^2 (e^{t/2})\sqrt{1 + \tfrac{1}{4}e^t}\, dt$$

→ set up problem.

25. Find the area inside one loop of the rose given by $r = \cos(4\theta)$.

$$A = \frac{1}{2}\int_{\cancel{a}\,-\pi/8}^{\cancel{b}\,\pi/8} f(\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/8}^{\pi/8} \cos^2(4\theta)\, d\theta$$

Bounds

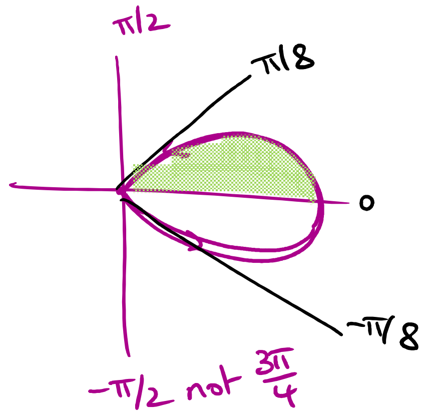
a & b solve for .

$r = 0$.

$\cos(4\theta) = 0$.

$4\theta = -\frac{\pi}{2}$ or $\frac{\pi}{2}$

$\theta = -\frac{\pi}{8}$ or $\frac{\pi}{8}$.



use symmetry .

$$A = \cancel{2} \cdot \frac{1}{\cancel{2}} \int_0^{\pi/8} \left[\frac{1 + \cos(8\theta)}{2} \right] d\theta$$

$$= \frac{1}{2}\theta \Big|_0^{\pi/8} + \frac{1}{2} \cdot \frac{\sin(8\theta)}{8} \Big|_0^{\pi/8}$$

$$= \frac{1}{2}\left(\frac{\pi}{8}\right) + \frac{1}{16}\Big(\underbrace{\sin\left(\cancel{8} \cdot \frac{\pi}{\cancel{8}}\right) - \sin(0)}_{0}\Big)$$

$$= \frac{\pi}{16} \quad \text{Ans.}$$