Week 11

Monday, November 4, 2024 7:26 PM

$\frac{0}{\#} = 0$ $\frac{5}{0} \to \pm\infty$

Math 151
Week-In-Review 11

4.4, 4.5, 4.7 (4.5 is not technically covered in Math 151, but it is a review of 4.3)

Todd Schrader

L' Hospital's
L' Hôpital's

Problem Statements

* Typo

1. Evaluate the following limits.

(a) $\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25} \overset{L'H}{=} \lim_{x\to 25} \frac{\frac{1}{2}x^{-1/2}}{1} = \lim_{x\to 25} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{25}} = \frac{1}{2(5)} = \boxed{\frac{1}{10}}$

$\frac{\sqrt{25}-5}{25-25} = \frac{0}{0}$ ✓

Use Algebra:

$\lim_{x\to 25} \frac{\cancel{\sqrt{x}-5}}{\cancel{(\sqrt{x}-5)}(\sqrt{x}+5)} = \lim_{x\to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{\sqrt{25}+5} = \frac{1}{10}$ ✓

(b) $\lim_{t\to 0} \frac{4^t-10^t}{t} \overset{L'H}{=} \lim_{t\to 0} \frac{4^t \cdot \ln(4) - 10^t \ln(10)}{1}$

$\frac{4^0-10^0}{0} = \frac{1-1}{0} = \frac{0}{0}$ ✓

$= 4^0 \ln(4) - 10^0 \ln(10) = \boxed{\ln(4)-\ln(10)}$

$(y^2+3)^{1/2}$

(c) $\lim_{y\to\infty} \frac{\sqrt{y^2+3}}{\sqrt{y^2-5}} \overset{L'H}{=} \lim_{y\to\infty} \frac{\frac{1}{2}(y^2+3)^{-1/2}\cdot 2y}{\frac{1}{2}(y^2-5)^{-1/2}\cdot 2y} = \lim_{y\to\infty} \frac{\sqrt{y^2-5}}{\sqrt{y^2+3}}$

$\frac{\sqrt{\infty^2+3}}{\sqrt{\infty^2-5}} = \frac{\infty}{\infty}$ ✓

$\lim_{y\to\infty} \frac{\sqrt{y^2\left(1+\frac{3}{y^2}\right)}}{\sqrt{y^2\left(1-\frac{5}{y^2}\right)}} = \lim_{y\to\infty} \frac{|y|\sqrt{1+3/y^2}}{|y|\sqrt{1-5/y^2}} = \lim_{y\to\infty} \frac{\cancel{y}\sqrt{1+3/y^2}}{\cancel{y}\sqrt{1-5/y^2}}$

$= \lim_{y\to\infty} \frac{\sqrt{1+3/y^2}}{\sqrt{1-5/y^2}} = \frac{\sqrt{1+3/\infty^2}}{\sqrt{1-5/\infty^2}} = \frac{\sqrt{1+0}}{\sqrt{1+0}} = \boxed{1}$

2. Evaluate the following limits.

(a) $\lim_{x\to\infty} x^{3/2}\sin(1/x)$ $= \lim_{x\to\infty} \frac{\sin(1/x)}{x^{-3/2}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\cos(1/x)\cdot(-x^{-2})}{-\frac{3}{2}x^{-5/2}}$

x^{-1}

$\infty^{3/2}\cdot\sin(1/\infty)$

$\infty\cdot\sin(0)$

$\infty\cdot 0$

$\frac{\sin(0)}{1/\infty} = \frac{0}{0}$ ✓

$= \lim_{x\to\infty} \frac{2}{3}\frac{x^{5/2}}{x^2}\cos(1/x) = \lim_{x\to\infty} \frac{2\sqrt{x}\cdot\cos(1/x)}{3}$

$= \frac{2\sqrt{\infty}\cdot\cos(1/\infty)}{3} = \infty\cdot\cos(0) = \boxed{\infty}$

(b) $\lim_{t\to 0}(\csc(t)-\cot(t))$ $= \lim_{t\to 0} \frac{1}{\sin(t)} - \frac{\cos(t)}{\sin(t)} = \lim_{t\to 0} \frac{1-\cos(t)}{\sin(t)}$

$\frac{1}{\sin(0)} - \frac{1}{\tan(0)}$

$\frac{1}{0} - \frac{1}{0}$

$\infty - \infty$

$\overset{L'H}{=} \lim_{t\to 0} \frac{\sin(t)}{\cos(t)} = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = \boxed{0}$

$\frac{1-\cos(0)}{\sin(0)} = \frac{0}{0}$ ✓

(c) $\lim_{y\to\infty} y^{1/y}$ $= L$

$\infty^{1/\infty}$

∞^{0}

$\ln\left(\lim_{y\to\infty} y^{1/y}\right) = \ln(L)$

$\lim_{y\to\infty} \ln\left(y^{1/y}\right) = \ln(L)$

$\lim_{y\to\infty} \frac{1}{y}\cdot\ln(y) = \ln(L)$

$\frac{1}{\infty}\cdot\ln(\infty)$

$0\cdot\infty$

$\ln(L) = \lim_{y\to\infty} \frac{\ln(y)}{y}$

$\frac{\ln(\infty)}{\infty} = \frac{\infty}{\infty}$ ✓

$\overset{L'H}{=} \lim_{y\to\infty} \frac{1/y}{1}$

$= \frac{1}{\infty} = 0$

$\ln(L) = 0$

$e^{\ln(L)} = e^{0}$

$\boxed{L = 1}$

3. Evaluate the following limits.

(a) $\lim_{x\to 1} \frac{x^x - 1}{\ln x + x - 1} \overset{L'H}{=} \lim_{x\to 1} \frac{(1+\ln x)x^x}{\frac{1}{x}+1} = \frac{(1+\ln 1)\cdot 1^1}{\frac{1}{1}+1} = \boxed{\frac{1}{2}}$

$\frac{1^1 - 1}{\ln(1)+1-1} = \frac{0}{0}$ ✓

$y = x^x$

$\ln(y) = \ln(x^x)$

$\ln(y) = x \cdot \ln(x)$

Take derivative both sides

$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln(x)\cdot 1$

$\frac{dy}{dx} = (1+\ln x)\cdot y$

$\frac{dy}{dx} = (1+\ln x)\cdot x^x$

Bonus:

$\lim_{x\to 0^+} \frac{x^x - 1}{\ln(x)+x-1} = 0$

$\frac{0^0 - 1}{\ln(0)+0-1} = \frac{\#}{-\infty} = 0$

(b) $\lim_{x\to\infty} \left(\frac{3x-2}{3x+7}\right)^{3x+1} = L$

$1^{3(\infty)+1} = 1^\infty$

$\ln(L) = \lim_{x\to\infty} (3x+1)\cdot\ln\left(\frac{3x-2}{3x+7}\right)$

$\infty \cdot \ln(1)$

$\infty \cdot 0$

$= \lim_{x\to\infty} \frac{\ln\left(\frac{3x-2}{3x+7}\right)}{(3x+1)^{-1}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{3x+7}{3x-2}\right)\cdot\left[9(3x+7)^{-2}\cdot 3\right]}{-(3x+1)^{-2}\cdot 3}$

$\frac{\ln(1)}{1/\infty} = \frac{0}{0}$ ✓

$\frac{3x+7-9}{3x+7}$

$\frac{3x+7}{3x+7} - \frac{9}{3x+7}$

$1 - \frac{9}{3x+7}$ ← $9(3x+7)^{-1}$

$\ln(L) = -9$

$\boxed{L = e^{-9}}$

4. Consider $f(x) = x^4 e^{-x}$. Find the following:

(a) Domain, Asymptotes, and Intercepts

Domain: $(-\infty, \infty)$ No V.A. H.A. $y = 0$

$$\lim_{x\to\infty} \frac{x^4}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{4x^3}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{12x^2}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{24x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{24}{e^x} = \frac{24}{\infty} = 0$$

$\frac{\infty}{\infty}$ ✓ $\frac{\infty}{\infty}$ ✓

$$\lim_{x\to-\infty} \frac{x^4}{e^x} = \frac{\infty}{e^{-\infty}} = \frac{\infty}{0} = \infty$$

Intercepts: y-int: $x=0$ $\frac{0}{e^0} = \frac{0}{1} = 0$ $\boxed{(0,0)}$ x-int: $y=0$ $0 = x^4 e^{-x}$

$x^4 = 0$ $e^{-x} = 0$

$x = 0$ N/A

(b) Intervals of Increase/Decrease, Locations of Local Extrema

$$f'(x) = x^4 \cdot e^{-x}(-1) + e^{-x} \cdot 4x^3 = x^3 e^{-x}(-x+4) = \frac{x^3(-x+4)}{e^x}$$

$f'(x) = e^{-x}(-x^4 + 4x^3)$

$f'(x)$ DNE: $e^x = 0$ N/A

$f'(x) = 0$: $\frac{x^3(-x+4)}{e^x} = 0$

$x^3(-x+4) = 0$ $x=0$, $x=4$

(−1) (1) (10)

(−)(+) 0 (+)(+) 4 (+)(−)

$f'(x)$: − + −

$f(x)$: ↘ ↗ ↘

Local Max: @ $x = 4$

Local Min: @ $x = 0$

(c) Intervals of Concavity, Locations of Inflection Points

$$f''(x) = e^{-x} \cdot (-4x^3 + 12x^2) + (-x^4 + 4x^3)\, e^{-x}(-1)$$
$$= -4x^3 e^{-x} + 12x^2 e^{-x} + x^4 e^{-x} - 4x^3 e^{-x}$$
$$= x^4 e^{-x} - 8x^3 e^{-x} + 12x^2 e^{-x}$$
$$= x^2 e^{-x}(x^2 - 8x + 12)$$
$$= \frac{x^2 (x-6)(x-2)}{e^x}$$

$f''(x)$ DNE: N/A

$f''(x) = 0$ $x=0$, $x=2$, $x=6$

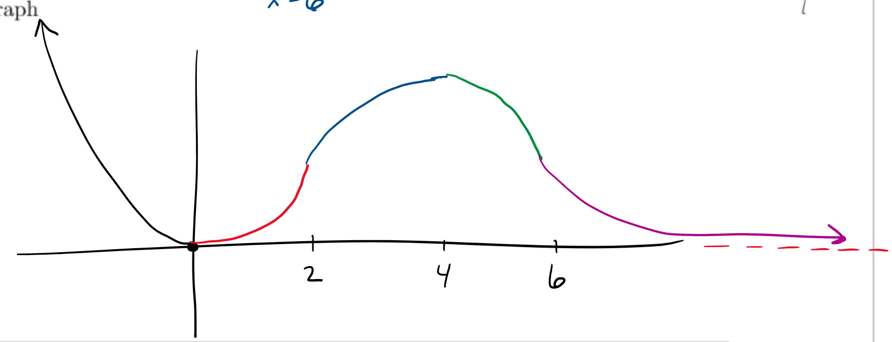
(−1) (1) (4) (10)

− − 0 − − 2 − + 6 + +

$f''(x)$: + + − +

$f(x)$: ∪ ∪ ∩ ∪

(d) Sketch a Graph



5. Consider $f(x) = \dfrac{\ln x}{x}$. Find the following:

(a) Domain, Asymptotes, and Intercepts

$x \neq 0$

$\ln(x)$ $x > 0$

Domain: $(0, \infty)$

Check: H.A. $\lim_{x\to\infty} f(x)$

V.A. $\lim_{x\to 0^+} f(x)$

(b) Intervals of Increase/Decrease, Locations of Local Extrema

(c) Intervals of Concavity, Locations of Inflection Points

(d) Sketch a Graph

$$\boxed{\left(\frac{19}{5},\ 3\left(\frac{19}{5}\right)-8\right)}$$

$y = 3x-8$

6. Find the point on the curve $f(x) = 3x - 8$ that is closest to $(2, 4)$.

$d = \sqrt{(x-2)^2 + (y-4)^2}$

Goal: Minimize Distance

Equivalently, Minimize $(\text{Distance})^2$

$d^2 = D = (x-2)^2 + (y-4)^2$

$D = (x-2)^2 + (3x-8-4)^2$

$D = (x-2)^2 + (3x-12)^2$

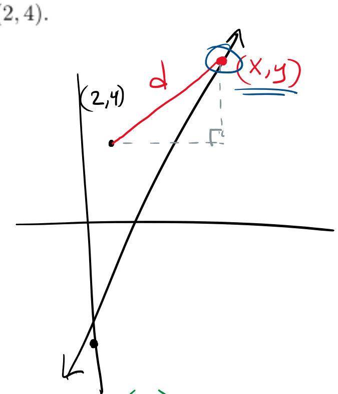
$D' = 2(x-2)\cdot 1 + 2(3x-12)\cdot(3)$

$= 2x-4 + 6(3x-12)$

$= 2x-4 + 18x - 72$

$D' = 20x - 76$

D' DNE : N/A $\quad D'=0 \quad 20x-76=0 \quad x = \frac{76}{20} = \frac{19}{5}$



D': (0) $-$ | $\frac{19}{5}$ | (1000) $+$

D: ↘ ↗

$2 < 5$

$2^2 < 5^2$

$\sqrt{2} < \sqrt{5}$

$\frac{1}{2} > \frac{1}{3}$

$\left(\frac{1}{2}\right)^2 > \left(\frac{1}{3}\right)^2$

7. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest from the point $(1, 0)$.

x-int: $(y=0)$ $\quad 4x^2 = 4 \quad x = \pm 1$

y-int: $x=0$ $\quad y^2 = 4 \quad y = \pm 2$

Maximize Distance:

$d = \sqrt{(x-1)^2 + (y-0)^2}$

Maximize $(\text{Distance})^2$

$y^2 = 4 - 4x^2$

$d^2 = D = (x-1)^2 + y^2$

$D = (x-1)^2 + 4 - 4x^2$

$D' = 2(x-1) - 8x$

$D' = 2x-2 - 8x$

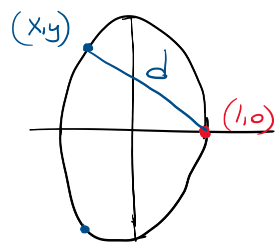
$D' = -6x - 2$

D' DNE: N/A

$D'=0$

$-6x-2=0 \quad -6x = 2$

$x = -\frac{2}{6} = -\frac{1}{3}$



D': (-1) $+$ | $-1/3$ | (0) $-$

D: ↗ ↘

$$\boxed{\begin{array}{c}\left(\frac{-1}{3},\ \frac{\sqrt{32}}{3}\right)\\ \left(\frac{-1}{3},\ \frac{-\sqrt{32}}{3}\right)\end{array}}$$

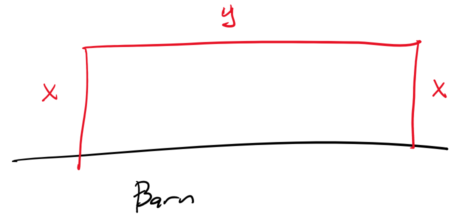
$x = \frac{-1}{3}$

$y^2 = 4 - 4\left(\frac{-1}{3}\right)^2 = 4 - \frac{4}{9}$

$y^2 = \frac{32}{9} \quad y = \pm\frac{\sqrt{32}}{3}$

8. A farmer wants to fence in a rectangular area of land adjacent to his barn. The area does not need fencing along the side of the barn. If the farmer has 1000 feet of fencing available, what is the largest area he can enclose?

Goal: Maximize Area: $A = x \cdot y$



$x + y + x = 1000$

$2x + y = 1000$

$y = 1000 - 2x$

$A = x \cdot (1000 - 2x) = 1000x - 2x^2$

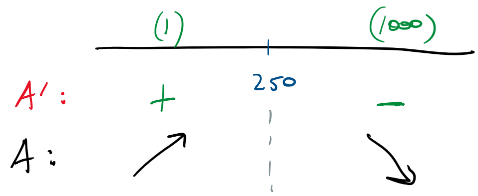
$A' = 1000 - 4x$

A' DNE : N/A

$A' = 0 \quad 1000 - 4x = 0$

$4x = 1000$

$x = 250$



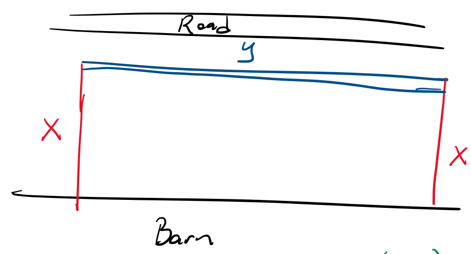
$y = 1000 - 2(250) = 500$

$\boxed{\text{Max Area: } (250)(500)}$

9. The farmer goes to install the fence, only to discover the wood he was going to use is no good, and he will need to purchase new fencing. He decides he just wants to buy enough fencing to enclose an area of 100,000 sq. ft. The barn runs parallel to a road, and he decides it would be good to purchase reinforcing fencing for the fence adjacent to the road. The reinforced fencing costs \$10 per foot, and the basic fencing cost \$8 per foot. What lengths of each fence should he purchase to enclose his land under these constraints for the least amount of money?

Minimize Cost:

$C = \underbrace{8(2x)}_{\text{Basic Fence}} + \underbrace{10(y)}_{\text{Reinforced}} = 16x + 10y$



Area: $xy = 100000 \Rightarrow y = \frac{100000}{x}$

$C = 16x + \frac{10 \cdot 100000}{x}$

$C' = 16 - \frac{1{,}000{,}000}{x^2}$

$C' = \frac{16x^2 - 1{,}000{,}000}{x^2} = 0$

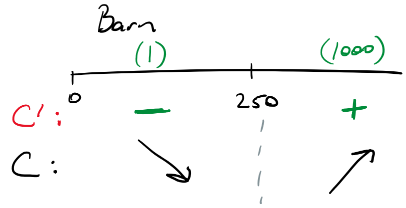
C' DNE: $x = 0$

$C' = 0$:

$16x^2 = 1{,}000{,}000$

$x^2 = \frac{1{,}000{,}000}{16}$

$x = \frac{1000}{4} = 250$



Basic Fence: $\boxed{500 \text{ ft.}}$

$y = \frac{100000}{250}$

$= 400$

Reinforced $\boxed{400 \text{ ft}}$

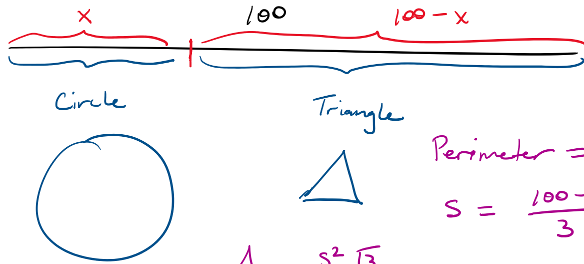
10. A cable is 100 ft long. It is to be cut into two pieces, and the two pieces will be fashioned into two shapes, an equilateral triangle and a circle. Determine the length of cable that should used on each shape such that the total area of the two shapes is as large as possible.

Circumference $= x$

$2\pi r = x$

$r = \frac{x}{2\pi}$

$A_{\circ} = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$



Perimeter $= 100 - x$

$s = \frac{100-x}{3}$

$A_{\Delta} = \frac{s^2\sqrt{3}}{4} = \left(\frac{100-x}{3}\right)^2 \cdot \frac{\sqrt{3}}{4}$

Maximize Total Area:

$$A = \frac{x^2}{4\pi} + \left(\frac{100-x}{3}\right)^2 \cdot \frac{\sqrt{3}}{4}$$

$$A' = \frac{2x}{4\pi} + \frac{2\sqrt{3}}{4}\left(\frac{100-x}{3}\right) \cdot \frac{-1}{3} = 0$$

A' DNE : N/A

$A' = 0$ $\quad \frac{x}{2\pi} - \frac{\sqrt{3}}{6} \cdot \frac{100-x}{3} = 0$

$$\frac{x}{2\pi} - \frac{\sqrt{3}}{6} \cdot \frac{100}{3} + \frac{\sqrt{3}}{6} \cdot \frac{x}{3}$$

11. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

$$x \cdot \frac{1}{2\pi} + x \cdot \frac{\sqrt{3}}{18} - \frac{100\sqrt{3}}{18} = 0$$

$$x\left(\frac{1}{2\pi} + \frac{\sqrt{3}}{18}\right) = \frac{100\sqrt{3}}{18}$$

$$x = \frac{\frac{100\sqrt{3}}{18}}{\frac{1}{2\pi} + \frac{\sqrt{3}}{18}} \approx 37.7$$

	(0)	37.7	(100)
A':	$-$		$+$
A:	$\searrow$		$\nearrow$

Max Area occurs if all 100 ft. of cable is used on Circle