Math 151 - Week-In-Review 2

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Topics for the week:

J.2 The Dot Product
J.3 Vector Functions and Parametric Curves
2.2 The Limit of a Function

J.2 The Dot Product

1. Given vectors $\mathbf{u} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = -2\mathbf{j}$, compute each of the following.

(a) $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$

$= 1(0) + 3(-2)$

$= 0 - 6$

$\vec{u} \cdot \vec{v} = -6$

Recall $\vec{u} = \vec{i} + 3\vec{j} = \langle 1, 3 \rangle$
$\vec{v} = -2\vec{j} = \langle 0, -2 \rangle$

The dot product is a scalar.

(b) $\left(-\frac{2}{7}\mathbf{v}\right) \cdot \left(\frac{4}{5}\mathbf{u}\right) = \left[-\frac{2}{7}(\vec{i} + 3\vec{j})\right] \cdot \left[\frac{4}{5}(0 - 2\vec{j})\right]$

$= \left\langle -\frac{2}{7}, -\frac{6}{7} \right\rangle \cdot \left\langle 0, -\frac{8}{5} \right\rangle$

$= \left(-\frac{2}{7}\right)(0) + \left(-\frac{6}{7}\right)\left(-\frac{8}{5}\right)$

$\left(-\frac{2}{7}\vec{v}\right)\left(\frac{4}{5}\vec{u}\right) = \frac{48}{35}$

(c) $(\mathbf{v} \cdot \mathbf{u})\mathbf{u} = [0(1) + (-2)(3)]\vec{u}$

$= (-6)\vec{u}$

$= (-6)(\vec{i} + 3\vec{j})$

$(\vec{v} \cdot \vec{u})\vec{u} = -6\vec{i} - 18\vec{j}$

The dot product times a vector is a vector.

(d) Compute the angle between vectors $\mathbf{u}$ and $\mathbf{v}$, leave your answer in exact form.

$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{u} \cdot \vec{v} = -6$ (see above)

$|\vec{u}| = \sqrt{(1)^2 + (3)^2} = \sqrt{1+9} = \sqrt{10}$

$|\vec{v}| = \sqrt{(0)^2 + (-2)^2} = \sqrt{4} = 2$

$\cos(\theta) = \frac{-6}{\sqrt{10} \cdot 2}$

$\boxed{\theta = \arccos\left(\frac{-3}{\sqrt{10}}\right) \quad \text{so } \frac{\pi}{2} < \theta < \pi}$

2. Given vectors $\mathbf{u} = \langle -5, 12\rangle$, $\mathbf{v} = \langle 6, 0\rangle$, and $\mathbf{w} = \left\langle -\frac{2}{5}, -\frac{3}{2}\right\rangle$, compute each of the following.

(a) $\mathbf{u} \cdot \mathbf{w} = u_1 w_1 + u_2 w_2$

$= (-5)\left(\frac{-2}{5}\right) + (12)\left(\frac{-3}{2}\right)$

$= 2 - 18$

$\vec{u} \cdot \vec{w} = -16$

(b) $2(\mathbf{v} \cdot \mathbf{u}) = 2\left[v_1 u_1 + v_2 u_2\right]$

$= 2\left[6(-5) + (0)(12)\right]$

$= 2\left[-30\right]$

$2(\vec{v} \cdot \vec{u}) = -60$

(c) $(2\mathbf{v}) \cdot \mathbf{u} = (2\langle 6, 0\rangle) \cdot \langle -5, 12\rangle$

$= \langle 12, 0\rangle \cdot \langle -5, 12\rangle$

$= (12)(-5) + (0)(12)$

$(2\vec{v}) \cdot \vec{u} = -60$

(d) The angle between vectors $\mathbf{v}$ and $\mathbf{w}$, leave your answer in exact form.

$\cos(\theta) = \dfrac{\vec{v} \cdot \vec{w}}{|\vec{v}|\,|\vec{w}|}$

$\vec{v} \cdot \vec{w} = 6 \cdot \left(-\frac{2}{5}\right) + 0\left(-\frac{3}{2}\right)$

$= -\frac{12}{5}$

$|\vec{v}| = \sqrt{(6)^2 + (0)^2} = 6$

$|\vec{w}| = \sqrt{\left(-\frac{2}{5}\right)^2 + \left(-\frac{3}{2}\right)^2} = \sqrt{\frac{4}{25} + \frac{9}{4}} = \frac{\sqrt{241}}{\sqrt{100}} = \frac{\sqrt{241}}{10}$

$\cos(\theta) = \dfrac{-12/5}{6 \cdot \frac{\sqrt{241}}{10}} = \dfrac{-12}{5} \cdot \dfrac{10}{6\sqrt{241}}$

$\boxed{\theta = \arccos\left(\dfrac{-4}{\sqrt{241}}\right)}$

(e) The orthogonal complement of $\mathbf{w}$

$\vec{w}_\perp = \langle -w_2, w_1\rangle$ or $\langle w_2, -w_1\rangle$

$\vec{w}_\perp = \left\langle \frac{3}{2}, \frac{-2}{5}\right\rangle$ or $\left\langle \frac{-3}{2}, \frac{2}{5}\right\rangle$

3. Determine the scalar product of two vectors, $\vec{a}$ and $\vec{b}$, if $\vec{a}$ has a length of 6 and $|\vec{b}| = 15$ and the smaller angle between the vectors is $\frac{3\pi}{4}$.

$|\vec{a}| = 6$

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos(\theta)$
$= (6)(15)\cos\left(\frac{3\pi}{4}\right)$
$= 90\cdot\left(-\frac{\sqrt{2}}{2}\right)$
$= -45\sqrt{2}$

The scalar product is the dot product of 2 vectors.

4. $\mathbf{u} = \left\langle -5, \frac{1}{2}\right\rangle$ and $\mathbf{v} = \langle 3, c\rangle$, determine the value of c such that the vectors are orthogonal.

$\vec{u}\cdot\vec{v} = 0$
$0 = (-5)(3) + \left(\frac{1}{2}\right)c$
$0 = -15 + \frac{1}{2}c$
$15 = \frac{1}{2}c$
$30 = c$

2 vectors are orthogonal if their dot product is 0.

5. Given vectors $\vec{u} = \left\langle \frac{3}{2}, -\frac{1}{2}\right\rangle$ and $\vec{w} = \langle 4, 8\rangle$, compute each of the following.

(a) Scalar projection of $\vec{u}$ onto $\vec{w}$

$\text{comp}_{\vec{w}}\vec{u} = \frac{\vec{u}\cdot\vec{w}}{|\vec{w}|}$
$= \frac{2}{4\sqrt{5}}$
$= \frac{1}{2\sqrt{5}}$

$\vec{u}\cdot\vec{w} = \left(\frac{3}{2}\right)(4) + \left(\frac{-1}{2}\right)(8)$
$= 6 - 4$
$= 2$

$|\vec{w}| = \sqrt{(4)^2 + (8)^2} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$

(b) Vector projection of $\vec{u}$ onto $\vec{w}$

$\text{proj}_{\vec{w}}\vec{u} = \frac{\vec{u}\cdot\vec{w}}{|\vec{w}|}\cdot\frac{\vec{w}}{|\vec{w}|}$
$= \frac{1}{2\sqrt{5}}\cdot\frac{\langle 4, 8\rangle}{4\sqrt{5}}$
$= \frac{1}{8\cdot 5}\langle 4, 8\rangle$
$= \left\langle \frac{1}{10}, \frac{1}{5}\right\rangle$

The vector projection is the vector with length equal to the scalar projection in the direction of $\vec{w}$.

need two points (0,5) and (3,4)

6. Determine the distance between the line, $3x + y = 5$ and the point $(4, 1)$, using vectors.

$\vec{a} = \langle 3,-4\rangle - \langle 0,5\rangle$
$= \langle 3,-9\rangle$ or $\langle 1,-3\rangle$
$\vec{a}_\perp = \langle 3,1\rangle$

$\vec{b} = \langle 4-0, 1-5\rangle$
$= \langle 4,-4\rangle$

$$\left|\text{comp}_{\vec{a}_\perp}\vec{b}\right| = \left|\frac{\vec{a}_\perp\cdot\vec{b}}{|\vec{a}_\perp|}\right|$$

distance is the scalar projection

$$= \left|\frac{(3)(4) + (1)(-4)}{\sqrt{(3)^2+(1)^2}}\right|$$

$$= \frac{8}{\sqrt{10}}$$

7. A kid pushes a sled 30 ft up a hill with an incline of 5°. The horizontal force exerted on the sled is 10 lbs. Determine the work done on the sled.

$W = |\vec{F}||\vec{D}|\cos(\theta)$

$W = 30(10)\cos(5°)$

$W = 300\cos(5°)$ ft-lbs
≈ 298.8584 ft-lbs

$|\vec{D}| = 30$ ft
$|\vec{F}| = 10$ lbs
$\theta = 5°$

8. An object is moved along a straight line from the point $(1, 7)$ to the point $(5, 16)$ at a constant force $\mathbf{F} = 14\mathbf{i} + 20\mathbf{j}$. Assuming the distance is measured in meters and the magnitude of the force is measured in newtons, determine each of the following:

(a) Displacement vector, $\mathbf{D}$.

$\vec{D} = (5,16) - (1,7)$
$= \langle 4,9\rangle$

or

$\vec{D} = 4\vec{i} + 9\vec{j}$

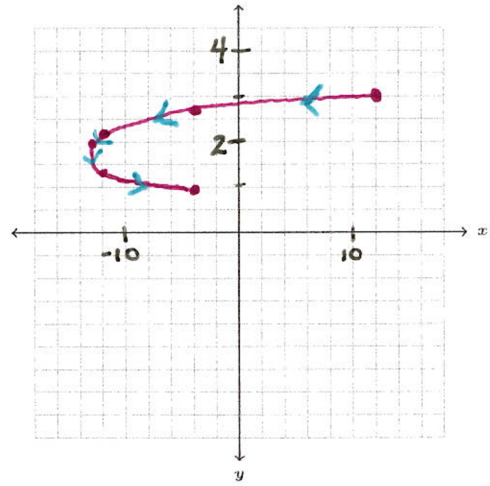
(b) Work, W.

$W = \vec{F}\cdot\vec{D}$
$= 4(14) + 9(20)$
$= 56 + 180$
$= 236$ Nm

J.3 Vector Functions and Parametric Curves

9. Sketch the curve generated by the parametric equations, $x = t^2 - 13$, $y = \sqrt{4-t}$, for $-5 \le t \le 3$. Then write the Cartesian equation for the the curve.

t	$x = t^2 - 13$	$y = \sqrt{4-t}$	
-5	12	3	$(12, 3)$
-3	-4	$\sqrt{7}$	$(-4, \sqrt{7})$
-1	-12	$\sqrt{5}$	$(-12, \sqrt{5})$
0	-13	2	$(-13, -2)$
1	-12	$\sqrt{3}$	$(-12, \sqrt{3})$
3	-4	1	$(-4, 1)$



$x = t^2 - 13$

$y = \sqrt{4-t}$ so $y^2 = 4 - t$ for $t \le 4$

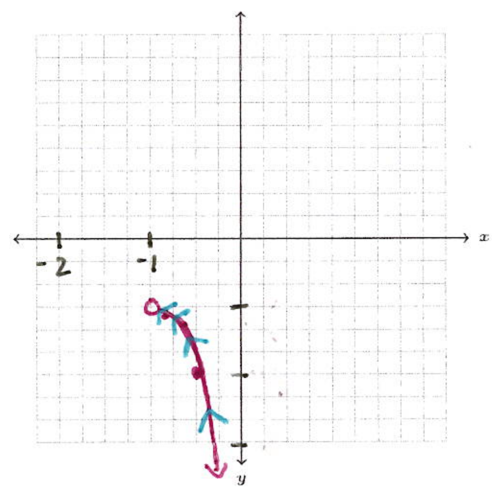
$y^2 - 4 = -t$ $\quad y \ge 0$

$4 - y^2 = t$

$x = (-y^2 + 4)^2 - 13$ $\quad$ 3 to 1 for y

10. Sketch the curve generated by the parametric equations, $x = \cos(t)$, $y = 3\sec t$, for $\frac{\pi}{2} < t < \pi$. Then write the Cartesian equation for the curve.

t	$x = \cos(t)$	$y = 3\sec(t)$	
close to $\frac{\pi}{2}$	0	undefined $\to -\infty$ from the right	
$\frac{2\pi}{3}$	$-1/2$	-6	$(-1/2, -6)$
$\frac{3\pi}{4}$	$-\sqrt{2}/2$	$\frac{-6}{\sqrt{2}}$	$(-\sqrt{2}/2, \frac{-6}{\sqrt{2}})$
$\frac{5\pi}{6}$	$-\sqrt{3}/2$	$\frac{-6}{\sqrt{3}}$	$(-\sqrt{3}/2, \frac{-6}{\sqrt{3}})$
close to π	-1	-3	$(-1, -3)$ hole



$x = \cos(t)$

$y = 3\sec(t) = \frac{3}{\cos(t)}$ $\quad \frac{\pi}{2} < t < \pi$

$y = \frac{3}{x}$ for $\quad$ 0 to -1 for x

11. Given the position of an object moving in the Cartesian plane is $\mathbf{r}(t) = \langle e^{2t}, e^{-t} \rangle$ after t-seconds, determine each of the following:

(a) The position of the object after 3 seconds.

$\vec{r}(t) = \langle e^{2t}, e^{-t} \rangle$

$e^{2(3)}$ and $e^{-(3)}$

$(x, y) = (e^{6}, e^{-3})$

(b) At what time is the object at the point $(1, 1)$?

Means

$e^{2t} = 1$ AND $e^{-t} = 1$ at the same t

$2t = \ln(1)$ $\qquad$ $-t = \ln(1)$

$2t = 0$ $\qquad$ $-t = 0$

$t = 0$ $\qquad$ $t = 0$

Same, so $t = 0$

(c) Does the object ever pass through the point $\left(\frac{1}{e^2}, e\right)$?

Means

$e^{2t} = \frac{1}{e^2}$ AND $e^{-t} = e$ at same t

$e^{2t} = e^{-2}$ $\qquad$ $e^{-t} = e^{1}$

$2t = -2$ $\qquad$ $-t = 1$

$t = -1$ $\qquad$ $t = -1$

Same, so $t = -1$ and yes

12. Write a vector equation of the line $y = 4x - 8$. (Hint: Use $(0, -8)$ as P_0.)

$\vec{r}(t) = \vec{r}_0 + \vec{r}t$

$= \langle 0, -8 \rangle + \langle 1, 4 \rangle t$

$= \langle 0 + 1t, -8 + 4t \rangle$

$= \langle 1t, -8 + 4t \rangle$

$\vec{r}_0 = \langle 0, -8 \rangle$

$\vec{r} = (1, -4) - (0, -8) = \langle 1, 4 \rangle$

we can note

$\vec{r} = \langle \Delta x, \Delta y \rangle$

and

$m = \frac{\Delta y}{\Delta x}$

13. Write a vector equation of the line perpendicular to the line $y = 6x - 7$ and passing through the point $(0, -7)$.

$\vec{r}(t) = \vec{r}_0 + \vec{r}\,t$

$\vec{r}_0 = \langle 0,-7\rangle$ $\quad m = \langle 1,6\rangle$

so

$\vec{r} = \langle -6, 1\rangle$

$= \langle 0,-7\rangle + \langle -6,1\rangle t$

$= \langle -6t, -7+t\rangle$

14. Write a parametric equation of the line passing through the points $(12, 5)$ and $(9, -2)$.

$\vec{r}(t) = \vec{r}_0 + \vec{r}\,t$

$\vec{r}_0 = \langle 12,5\rangle$

$\vec{r} = \langle 9-12, -2-5\rangle = \langle -3, -7\rangle$

$= \langle 12,5\rangle + \langle -3,-7\rangle t$

$= \langle 12-3t, 5-7t\rangle$

$x(t) = 12-3t$

$y(t) = 5-7t$

15. Determine the parametric equations for the line that passes through the point (3,-1) and is

(a) is parallel to the vector $\langle -5, -4\rangle$.

$\vec{r}(t) = \langle 3,-1\rangle + \langle -5,-4\rangle t$

$= \langle 3-5t, -1-4t\rangle$

$x(t) = 3-5t$

$y(t) = -1-4t$

(b) is perpendicular to the vector $\langle -5, -4\rangle$. $\quad a_\perp = \langle 4,-5\rangle$

$\vec{r}(t) = \langle 3,-1\rangle + \langle 4,-5\rangle t$

$= \langle 3+4t, -1-5t\rangle$

$x = 3+4t$

$y = -1-5t$

16. State the slope of the line with corresponding vector equation $\mathbf{r}(t) = \langle 5 - 2t, -8 + 7t \rangle$.

$\vec{r}(t) = \langle 5 - 2t, -8 + 7t \rangle$

$= \langle 5, -8 \rangle + \langle -2, 7 \rangle t$

the slope is based or $\vec{r}$

$m = \frac{7}{-2} = -\frac{7}{2}$

17. Determine whether the lines, $L_1 = \mathbf{r}(t) = (-6 + 2t)\mathbf{i} + (7 - 6t)\mathbf{j}$ and $L_2 = \mathbf{r}(s) = \left(5 + \frac{1}{2}s\right)\mathbf{i} + \left(-8 + \frac{3}{2}s\right)\mathbf{j}$, are parallel, perpendicular, or neither.

$\vec{r}(t) = \langle -6, 7 \rangle + \langle 2, -6 \rangle t$ $\qquad m_1 = \frac{-6}{2} = -3$

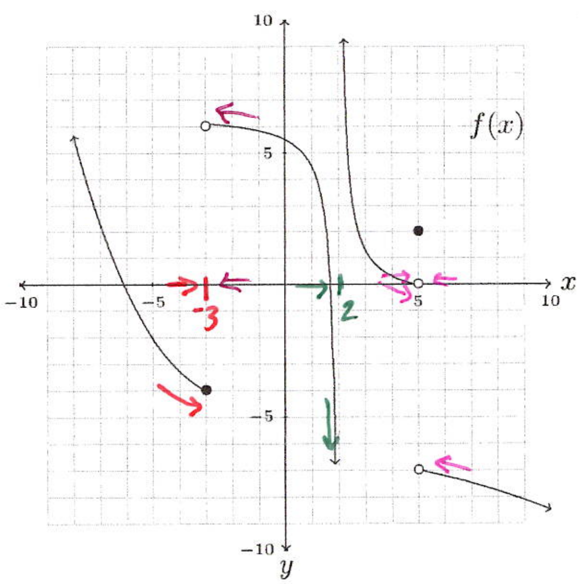
$\vec{r}(s) = \langle 5, -8 \rangle + \langle \frac{1}{2}, \frac{3}{2} \rangle s$ $\qquad m_2 = \frac{3/2}{1/2} = 3$

$m_1 \neq m_2$ so L_1 and L_2 are not parallel

$m_1 \cdot m_2 \neq -1$ so L_1 and L_2 are not perpendicular

2.2 The Limit of a Function

18. Use the graph provided to answer each of the following:



a. $\lim_{x \to -3^-} f(x) = -4$

b. $\lim_{x \to -3^+} f(x) = 6$

c. $\lim_{x \to 2^-} f(x) = -\infty$

d. $\lim_{x \to 5} f(x)$ does not exist

19. If $\lim_{x\to 4^-} f(x) = -25$ and $\lim_{x\to 4^+} f(x) = -26$, what can we say about $\lim_{x\to 4} f(x)$?

$\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$ so $\lim_{x\to 4} f(x)$ does not exist

20. If $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = -\infty$, what can we say about the function $f(x)$ at $x = 4$ and the graph of $f(x)$ at $x = 4$.

f(x) is undefined at x = 4, the graph of f(x) has a vertical asymptote at x = 4

21. Evaluate each of the following limits.

(a) $\lim_{x\to -8^+} \left(\frac{\overbrace{x^2+3x+1}^{\to 41}}{\underbrace{8+x}_{\to 0^+}} \right) = \infty$

$(-8)^2 + 3(-8) + 1 = 64 - 24 + 1 = 4$

$8 + (-8) = 0$

$\to -8^+$ means $-7.9, -7.99, -7.999$ so $x + 8 > 0$

(b) $\lim_{x\to 3^-} \left(\frac{\overbrace{1-x}^{\to -2}}{\underbrace{2x-6}_{\to 0^-}} \right) = \infty$

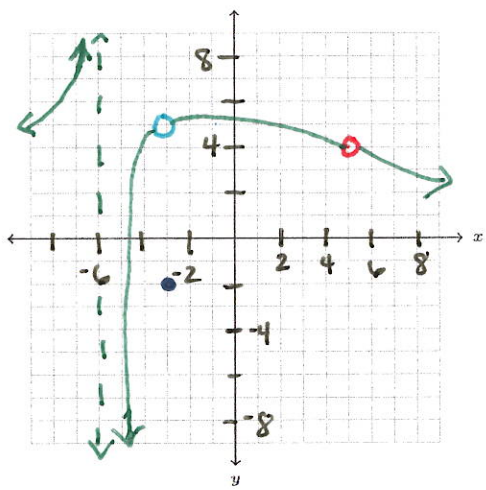
$1 - (3) = -2$

$2(3) - 6 = 6 - 6 = 0$

$\to 3^-$ means $2.9, 2.99, 2.999$ so $2x - 6 < 0$

22. Sketch the graph of a function, $g(x)$, that satisfy the conditions below:

- $g(-3) = -2$ point at $(-3,-2)$
- $g(5)$ is undefined hole or vertical asymptote
- $\lim_{x\to -6^-} g(x) = \infty$
- $\lim_{x\to -6^+} g(x) = -\infty$ } vertical asymptote at $x=-6$
- $\lim_{x\to -3^-} g(x) = 5$
- $\lim_{x\to -3^+} g(x) = 5$ } hole at $(-3,5)$
- $\lim_{x\to 5} g(x) = 4$ - hole at $(5,4)$



23. Sketch the graph of $h(x) = \dfrac{x}{(3x-12)^2}$.

$3x - 12 = 0$

$3x = 12$

$x = 4$ is a vertical asymptote

$$\lim_{x\to 4^-}\left[\frac{\overbrace{x}^{\to 4}}{\underbrace{(3x-12)^2}_{\to 0^+}}\right] = \infty$$

$$\lim_{x\to 4^+}\left[\frac{\overbrace{x}^{\to 4}}{\underbrace{(3x-12)^2}_{\to 0^+}}\right] = \infty$$

x-intercept: $(0,0)$