Week 3

Monday, September 9, 2024 7:29 PM

Math 151
Week-In-Review 3

~~2.4~~ 2.3 and 2.5

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Problem Statements

Section 2.3

1. Evaluate the following limits.

$= 2(8^2) - 10(8) + 5 = \boxed{53}$

(a) $\lim_{x \to 8} (2x^2 - 10x + 5) = \lim_{x \to 8} 2x^2 - \lim_{x \to 8} 10x + \lim_{x \to 8} 5$

$= 2 \underbrace{\lim_{x \to 8}}_{} x^2 - 10 \underbrace{\lim_{x \to 8} x}_{} + \underbrace{\lim_{x \to 8} 5}_{}$

$= 2(64) - 10(8) + 5 = 128 - 80 + 5 = \boxed{53}$

(b) $\lim_{x \to 5^+} \frac{1}{x-5} = \boxed{\infty}$ $\quad f(x) = \frac{1}{x-5}$ $\quad f(5)$ DNE

$\frac{1}{5-5} = \frac{1}{0}$ ← non-zero ← zero

$x = 5.1$ $\quad \frac{1}{5.1 - 5} = \frac{1}{0.1^+}$

?

$\frac{(x+5)(x-5)}{(x-5)} = \frac{x+5}{1}$

$(x \neq 5)$ $\quad$ if $x \neq 5$

(c) $\lim_{x \to 5^+} \frac{x^2 - 25}{(x-5)^2}$ $\quad f(x) = \frac{x^2-25}{(x-5)^2}$ $\quad f(5)$ DNE

$\frac{25-25}{(5-5)^2} = \frac{0}{0}$

$\lim_{x \to 5^+} \frac{(x+5)\cancel{(x-5)}}{(x-5)\cancel{(x-5)}} = \lim_{x \to 5^+} \frac{x+5}{x-5} = \boxed{\infty}$

$\frac{5+5}{5-5} = \frac{10}{0}$ $\quad x = 5.1$ $\quad \frac{10}{5.1-5} = \frac{10}{0.1^+}$

(d) $\lim_{h\to 0} \frac{(5+h)^2-5^2}{h} = \lim_{h\to 0} \frac{(25+10h+h^2)-25}{h}$

$\frac{(5+0)^2-5^2}{0} = \frac{0}{0}$

$= \lim_{h\to 0} \frac{10h+h^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(10+h)}{\cancel{h}}$

$= \lim_{h\to 0} 10+h = 10+0 = \boxed{10}$

(e) $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{(\sqrt{t^2+9}+3)}{(\sqrt{t^2+9}+3)}$

$\sqrt{t^2+9} \quad (\sqrt{t^2+9}+3)$

$-3 \quad (\sqrt{t^2+9}+3)$

$\frac{\sqrt{0^2+9}-3}{0^2} = \frac{0}{0}$

First Outer Inner Last

$= \lim_{t\to 0} \frac{\sqrt{t^2+9}\sqrt{t^2+9} + \cancel{\sqrt{t^2+9}\cdot 3 - 3\sqrt{t^2+9}} - 3(3)}{t^2(\sqrt{t^2+9}+3)}$

$= \lim_{t\to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$

$= \frac{1}{\sqrt{0^2+9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$

$$\frac{3}{3+3} + \frac{\sqrt{7-3}-2}{\sqrt{4-3}-1} = \frac{3}{6} + \frac{0}{0}$$

(f) $\lim_{x\to 3}\left(\frac{3}{x+3} + \frac{\sqrt{7-x}-2}{\sqrt{4-x}-1}\right) = \lim_{x\to 3}\frac{3}{x+3} + \lim_{x\to 3}\frac{\sqrt{7-x}-2}{\sqrt{4-x}-1} = \boxed{1}$

$$\lim_{x\to 3}\frac{3}{x+3} = \frac{3}{6} = \frac{1}{2}$$

$$\lim_{x\to 3}\frac{\sqrt{7-x}-2}{\sqrt{4-x}-1}\cdot\frac{(\sqrt{4-x}+1)}{(\sqrt{4-x}+1)} = \lim_{x\to 3}\frac{(\sqrt{7-x}-2)(\sqrt{4-x}+1)}{(4-x)-1}$$

$$= \lim_{x\to 3}\frac{(\sqrt{4-x}+1)(\sqrt{7-x}-2)}{3-x}\cdot\frac{(\sqrt{7-x}+2)}{(\sqrt{7-x}+2)} = \lim_{x\to 3}\frac{(\sqrt{4-x}+1)(7-x-4)}{(3-x)(\sqrt{7-x}+2)}$$

$$= \lim_{x\to 3}\frac{(\sqrt{4-x}+1)\cancel{(3-x)}}{\cancel{(3-x)}(\sqrt{7-x}+2)} = \lim_{x\to 3}\frac{\sqrt{4-x}+1}{\sqrt{7-x}+2} = \frac{\sqrt{4-3}+1}{\sqrt{7-3}+2} = \frac{2}{4} = \frac{1}{2}$$

2. Find $\lim_{x\to -4} f(x)$ and $\lim_{x\to 0} f(x)$ for the function below.

$$f(x) = \begin{cases} 5-x & \text{if} \quad x < -4 \\ x+13 & \text{if} \quad -4 \le x \le 0 \\ 12 & \text{if} \quad x > 0 \end{cases}$$

$f(-2) = -2+13$

$f(1) = 12$

$f(0) =$

$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} 5-x = 5-(-4) = 9$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} x+13 = -4+13 = 9$

$\boxed{\lim_{x\to -4} f(x) = 9}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+13 = 0+13 = 13$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 12 = 12$

$\boxed{\lim_{x\to 0} f(x) \text{ DNE}}$

3. Find $\lim_{x\to 2} g(x)$ for the function below.

$$g(x) = \begin{cases} \dfrac{\frac{8}{x} - 4}{x-2} & \text{if } x < 2 \\ x+3 & \text{if } x = 2 \\ \dfrac{x^2+2x-8}{x^2-7x+10} & \text{if } x > 2 \end{cases}$$

$g(2) = 2+3 = 5$

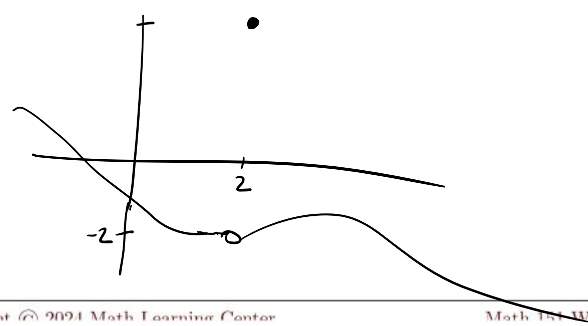
$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} \frac{\frac{8}{x} - 4\left(\frac{x}{x}\right)}{x-2} = \lim_{x\to 2^-} \frac{\frac{8}{x} - \frac{4x}{x}}{\left(\frac{x-2}{1}\right)} = \lim_{x\to 2^-} \frac{8-4x}{x} \cdot \frac{1}{(x-2)}$$

$$= \lim_{x\to 2^-} \frac{4(2-x)}{x(x-2)} = \lim_{x\to 2^-} \frac{-4\cancel{(x-2)}}{x\cancel{(x-2)}} = \lim_{x\to 2^-} \frac{-4}{x} = \frac{-4}{2} = -2$$

$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \frac{x^2+2x-8}{x^2-7x+10} = \lim_{x\to 2^+} \frac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}(x-5)}$$

$$= \lim_{x\to 2^+} \frac{x+4}{x-5} = \frac{2+4}{2-5} = \frac{6}{-3} = -2$$

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) \qquad \boxed{\lim_{x\to 2} g(x) = -2}$$



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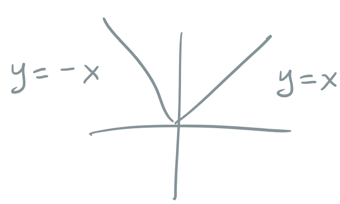
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



4. Evaluate the following limits.

$$|x-7| = \begin{cases} x-7 & \text{if } x-7 \geq 0 \\ -(x-7) & \text{if } x-7 < 0 \end{cases}$$

$x \geq 7$

$x < 7$

(a) $\lim_{x\to 7^-} \dfrac{14-2x}{|x-7|}$

$$\lim_{x\to 7^-} \frac{14-2x}{|x-7|} = \lim_{x\to 7^-} \frac{2(7-x)}{-(x-7)} = \lim_{x\to 7^-} \frac{-2\cancel{(x-7)}}{-\cancel{(x-7)}} = \lim_{x\to 7^-} 2 = \boxed{2}$$

(b) $\lim_{x\to 7^+} \dfrac{14-2x}{|x-7|}$

$$\lim_{x\to 7^+} \frac{14-2x}{|x-7|} = \lim_{x\to 7^+} \frac{-2\cancel{(x-7)}}{\cancel{x-7}} = \lim_{x\to 7^+} -2 = \boxed{-2}$$

$$\lim_{x\to 7^+} \frac{14-2x}{|x-7|} = \lim_{x\to 7^+} \frac{-2\cancel{(x-7)}}{\cancel{x-7}} = \lim_{x\to 7^+} -2 = \boxed{-2}$$

(c) $\lim_{x\to 7} \frac{14-2x}{|x-7|}$ DNE because $\lim_{x\to 7^-} f(x) \neq \lim_{x\to 7^+} f(x)$

$\frac{1}{0} - \frac{1}{0}$?

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$x = -0.1$

$\frac{2}{-0.1}$

(d) $\lim_{x\to 0^-} \frac{1}{x} - \frac{1}{|x|}$

$$= \lim_{x\to 0^-} \frac{1}{x} - \frac{1}{-x} = \lim_{x\to 0^-} \frac{1}{x} + \frac{1}{x} = \lim_{x\to 0^-} \frac{2}{x} = \boxed{-\infty}$$

(e) $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{|x|}$

$$= \lim_{x\to 0^+} \frac{1}{x} - \frac{1}{x} = \lim_{x\to 0^+} \frac{1-1}{x} = \lim_{x\to 0^+} \frac{0}{x} = \lim_{x\to 0^+} 0 = \boxed{0}$$

(f) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{|x|}$ DNE

5. Prove $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0.$

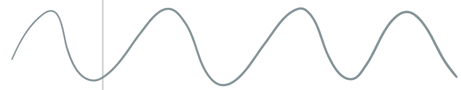
Squeeze Theorem

$\sqrt{0} \cdot e^{\sin(\pi/0)}$

$0 \cdot e^{\sin(\infty)}$

$0 \cdot e^{DNE}$

$0 \cdot (DNE)$



$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$$e^{-1} \leq e^{\sin\left(\frac{\pi}{x}\right)} \leq e^{1}$$

$$\sqrt{x} \cdot e^{-1} \leq \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)} \leq \sqrt{x} \cdot e$$

$$\lim_{x\to 0^+} \sqrt{x}\, e^{-1} \leq \lim_{x\to 0^+} \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)} \leq \lim_{x\to 0^+} \sqrt{x} \cdot e$$

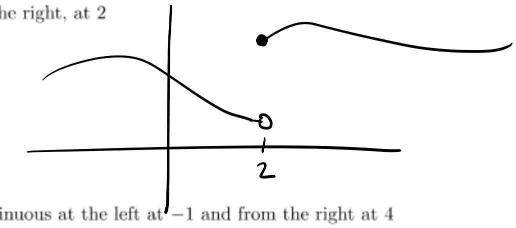
$$0 \leq \lim_{x\to 0^+} \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)} \leq 0$$

$$\boxed{\lim_{x\to 0^+} \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)} = 0 \text{ by Squeeze Theorem}}$$

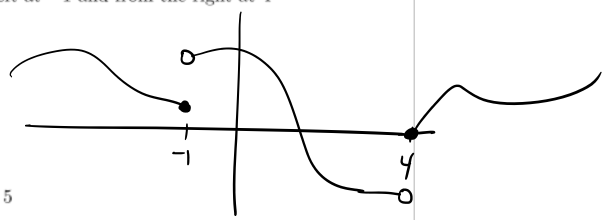
Section 2.5

6. Sketch the graph of a function that is continuous except for at the stated discontinuities.

   (a) Discontinuous, but continuous from the right, at 2

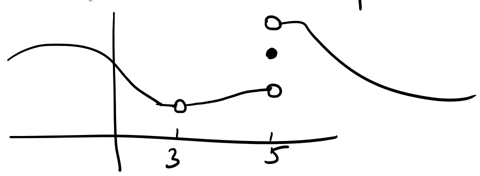


   (b) Discontinuities at −1 and 4, but continuous at the left at −1 and from the right at 4

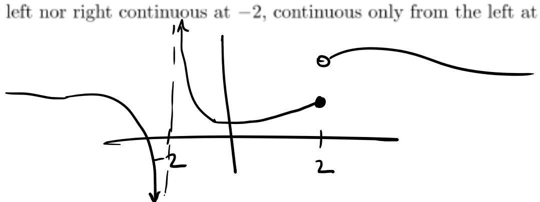


   (c) Removable discontinuity at 3, jump discontinuity at 5

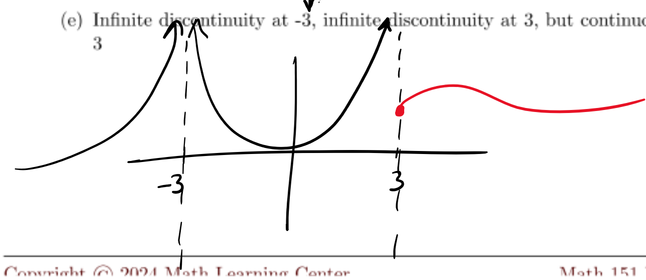
   Hole



   (d) Neither left nor right continuous at −2, continuous only from the left at 2



   (e) Infinite discontinuity at -3, infinite discontinuity at 3, but continuous from the right at 3



7. Prove $f(x)$ is continuous at 5 for the following function.

$$f(x) = \begin{cases} \sqrt{14-x} & \text{if } x < 5 \\ 8-x & \text{if } x = 5 \\ 23-4x & \text{if } x > 5 \end{cases}$$

1 $f(5) = 8-5 = 3$

2 $\lim_{x\to 5} f(x)$ exists

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \sqrt{14-x} = \sqrt{14-5}$

$= \sqrt{9} = 3$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} 23-4x = 23-4(5) = 3$

$\lim_{x\to 5} f(x) = 3$

3 $\lim_{x\to 5} f(x) \overset{?}{=} f(5)$ ✓

Therefore $f(x)$ is continuous @ $x=5$

8. Determine the values of x for which the following function is discontinuous.

$$f(x) = \begin{cases} -12 - x & \text{if} \quad x \le -4 \\ \dfrac{x^2 - 16}{x^2 + 9x + 20} & \text{if} \quad -4 < x \le 4 \\ \dfrac{x+5}{x-5} & \text{if} \quad x > 4 \end{cases}$$

$f(-5) =$

Discontinuities @ $x = 4$ and @ $x = 5$

$-12 - x$ $\qquad \dfrac{x^2-16}{x^2+9x+20}$ $\qquad \dfrac{x+5}{x-5}$

$-5 \quad -4 \quad 4 \quad 5$

✓

$x^2 + 9x + 20 \neq 0$

$(x+5)(x+4) \neq 0$

$x+5 \neq 0 \quad x+4 \neq 0$

$x \neq -5 \quad x \neq -4$

Not an issue

$x - 5 \neq 0$

$x \neq 5$

@ $x = -4$

① $f(-4) = -12 - (-4) = -8$

② $\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} -12 - x = -8$

$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} \dfrac{x^2-16}{x^2+9x+20}$

$= \lim_{x \to -4^+} \dfrac{(x-4)\cancel{(x+4)}}{\cancel{(x+4)}(x+5)} = \lim_{x \to -4^+} \dfrac{x-4}{x+5}$

$= \dfrac{-4-4}{-4+5} = \dfrac{-8}{1} = -8$

$\lim_{x \to -4} f(x) = -8$

③ $\lim_{x \to -4} f(x) = f(-4)$ ✓

Continuous @ $x = -4$

@ $x = 4$

① $f(4) = \dfrac{4^2 - 16}{4^2 + 36 + 20}$

$= \dfrac{0}{72} = 0$

② $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \dfrac{x^2-16}{x^2+9x+20} = 0$

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \dfrac{x+5}{x-5} = \dfrac{9}{-1} = -9$

$\lim_{x \to 4} f(x)$ DNE

9. Determine the values of a and b that make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} x+3 & \text{if} \quad x < a \\ x^2 - 8x - 7 & \text{if} \quad a \le x < 5 \\ -4x + b & \text{if} \quad x \ge 5 \end{cases}$$

$x+3$ | x^2-8x-7 | $-4x+b$

a $\qquad$ 5

✓ ✓ ✓

Polynomials are continuous everywhere

$x = a$

① $f(a) = a^2 - 8a - 7$

② $\lim_{x\to a} f(x)$ exists ✓

$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} x+3 = a+3$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} x^2-8x-7 = a^2-8a-7$

$a+3 = a^2-8a-7$

$0 = a^2 - 9a - 10$

$(a-10)(a+1) = 0$

$\cancel{a=10}$ $\quad \boxed{a=-1}$

③ $\lim_{x\to -1} f(x) = f(-1)$ ✓

@ $x = 5$ ① $f(5) = -4(5) + b$

$= -20 + b$

② $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} x^2 - 8x - 7$

$= 25 - 40 - 7 = -15 - 7 = -22$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} -4x+b = -20+b$

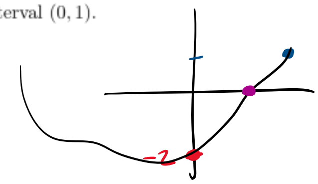
$-22 = -20 + b$

$\boxed{-2 = b}$

③ $\lim_{x\to 5} f(x) = f(5)$ ✓

10. Prove there is a solution to the equation $e^x = 3 - 2x$ on the interval $(0, 1)$.

Intermediate Value Theorem.



$$\underbrace{e^x - 3 + 2x}_{} = 0$$

$$f(x) = e^x - 3 + 2x$$

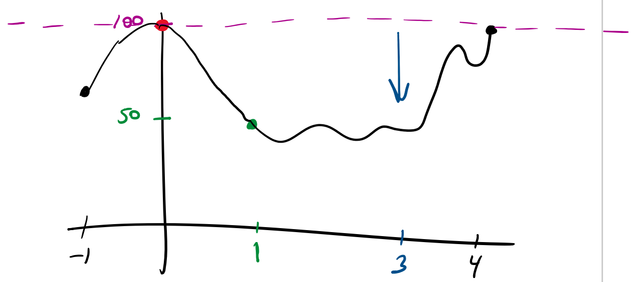
$$f(0) = e^0 - 3 + 2(0) = 1 - 3 + 0$$
$$= -2$$

$$e \approx 2.7$$

$$f(1) = e^1 - 3 + 2(1) = e + 2 - 3 = e - 1 > 0$$

Because $f(x)$ is continuous $[0,1]$ and $f(0) < 0$ and $f(1) > 0$, there must exist a value c in the interval $(0,1)$ such that $f(c) = 0$ by the Intermediate Value Theorem (IVT).

11. Suppose f is continuous on $[-1, 4]$ and the only solutions to $f(x) = 100$ are $x = 0$ and $x = 4$. If $f(1) = 50$, explain why $f(3) < 100$.



If $f(3) > 100$ then the IVT would tell us there is a solution to $f(x) = 100$ on the interval $(1,3)$.

We know the only solutions to $f(x) = 100$ are $x = 0$ and $x = 4$, so $f(3) < 100$.