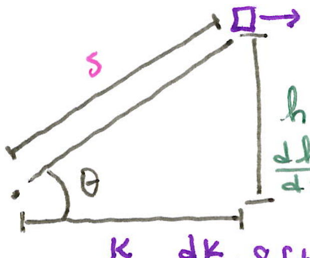
Math 151 - Week-In-Review 8

Topics for the week:

3.9 Related Rates
Review for Exam 2 (3.1 - 3.9)

3.9 Related Rates

1. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal changing when 200 ft of string have been let out?



$h = 100\,ft$

$\frac{dh}{dt} = 0\,ft/s$

$\frac{dK}{dt} = 8\,ft/s$

$\frac{d\theta}{dt} = ?$

$s = 200$

$(200)^2 = (100)^2 + K^2$

$30000 = K^2$

$100\sqrt{3} = K$

$\cot(\theta) = \frac{K}{100}$ ← because this is constant

$-\csc^2(\theta)\frac{d\theta}{dt} = \frac{1}{100}\frac{dK}{dt}$

$-\left(\frac{200}{100}\right)^2\frac{d\theta}{dt} = \frac{1}{100}\cdot 8$

$-4\frac{d\theta}{dt} = \frac{2}{25}$

$\frac{d\theta}{dt} = \frac{-1}{50}$ rad/s

2. A point moves around the ellipse $4x^2 + 9y^2 = 75$. When the point is at $\left(\sqrt{3}, \sqrt{7}\right)$, its x-coordinate is increasing at a rate of 10 units per second. What is the rate of change of the y-coordinate at that instant?

$4x^2 + 9y^2 = 75$

$x = \sqrt{3}$ $\quad \frac{dx}{dt} = 10$ u/s

$y = \sqrt{7}$ $\quad \frac{dy}{dt} = ?$

$4x^2 + 9y^2 = 75$

$8x\frac{dx}{dt} + 18y\frac{dy}{dt} = 0$

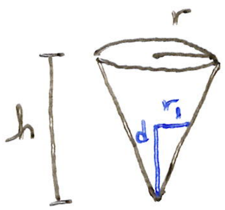
$8(\sqrt{3})(10) + 18(\sqrt{7})\frac{dy}{dt} = 0$

$80\sqrt{3} + 18\sqrt{7}\frac{dy}{dt} = 0$

$18\sqrt{7}\frac{dy}{dt} = -80\sqrt{3}$

$\frac{dy}{dt} = \frac{-80\sqrt{3}}{18\sqrt{7}} = \frac{-40\sqrt{3}}{9\sqrt{7}}$ u/s

3. A water tank has the shape of an inverted right circular cone of altitude 18 ft and a base radius of 6 ft. If water is being pumped into the tank at a rate of 10 gal/min(≈ 1.337 ft^3/min), find the rate at which the water level is rising when the water is 5 ft deep.



$h = 18$ ft
$r = 6$ ft

$$V = \frac{1}{3}\pi r_1^2 d$$

$$V = \frac{1}{3}\pi \left(\frac{1}{3}d\right)^2 d$$

$$V = \frac{1}{27}\pi d^3$$

$\frac{dV}{dt} = 1.337$ ft^3/min

$d = 5$ ft $\qquad \frac{dd}{dt} = ?$

$$\frac{dV}{dt} = \frac{1}{9}\pi d^2 \cdot \frac{dd}{dt}$$

$$1.337 \approx \frac{1}{9}\pi (5)^2 \cdot \frac{dd}{dt}$$

$$1.337 \approx \frac{25\pi}{9} \cdot \frac{dd}{dt}$$

$$\frac{1.337 \cdot 9}{25\pi} \text{ ft/min} = \frac{dd}{dt}$$

$$0.1532 \text{ ft/min} = \frac{dd}{dt}$$

Note:

$$\frac{h}{r} = \frac{d}{r_1}$$

$$\frac{18}{6} = \frac{d}{r_1}$$

$$r_1 = \frac{1}{3}d$$

Review for Exam 2

4. Determine values of x where the function $h(x) = \dfrac{2x}{\sqrt{x^3+4}}$ has a horizontal tangent.

$\frac{dh}{dx} = 0$

$$h(x) = \frac{2x}{(x^3+4)^{1/2}}$$

Note: $x^3 + 4 > 0$

$$h'(x) = \frac{2(x^3+4)^{1/2} - \frac{1}{2}(x^3+4)^{-1/2}(3x^2)\cdot 2x}{\left((x^3+4)^{1/2}\right)^2}$$

$$h'(x) = \left(\frac{2(x^3+4)^{1/2} - 3x^3(x^3+4)^{-1/2}}{x^3+4}\right)\frac{(x^3+4)^{1/2}}{(x^3+4)^{1/2}} = \frac{2(x^3+4) - 3x^3}{(x^3+4)^{3/2}}$$

$0 = 2x^3 + 8 - 3x^3$

$0 = -x^3 + 8$

$-8 = -x^3$

$+2 = x$

So $h(x)$ has a horizontal tangent at $x = 2$.

5. Compute the derivative of $f(x) = x^4 \sin(x)$.

$f(x) = x^4 \cdot \sin(x)$

$f'(x) = 4x^3 \sin(x) + \cos(x) \cdot x^4$

$f'(x) = 4x^3 \sin(x) + x^4 \cos(x)$

6. Compute the derivative of $g(x) = (6 - 3x^2)^5$.

$g(x) = (6 - 3x^2)^5$

$\frac{dg(x)}{dx} = 5(6 - 3x^2)^4 \cdot (-6x)$

$\frac{dg(x)}{dx} = -30x\,(6 - 3x^2)^4$

7. Compute the derivative of $h(x) = \arctan\left(x^3 - \sqrt{9-x}\right)$.

$$h(x) = \arctan\left(x^3 - (9-x)^{1/2}\right)$$

$$\frac{dh(x)}{dx} = \frac{1}{1+\left(x^3-(9-x)^{1/2}\right)^2} \cdot \left(3x^2 - \frac{1}{2}(9-x)^{-1/2}(-1)\right)$$

$$\frac{dh(x)}{dx} = \frac{6x^2(9-x)^{1/2}+1}{2(9-x)^{1/2}\left(1+\left(x^3-(9-x)^{1/2}\right)^2\right)}$$

8. Compute the derivative of $f(x) = \cos^2\left(12x^{-5}\right)$.

$$f(x) = \left(\cos(12x^{-5})\right)^2$$

$$f'(x) = 2\left(\cos(12x^{-5})\right) \cdot \left(-\sin(12x^{-5}) \cdot (-60x^{-6})\right)$$

$$f'(x) = 60x^{-6}\left(2\cos(12x^{-5})\sin(12x^{-5})\right)$$

$$f'(x) = \frac{60\sin(24x^{-5})}{x^6}$$

$2\sin(\theta)\cos(\theta) = \sin(2\theta)$

9. Compute the first and second derivatives of $f(x) = \ln(5x^2 - 4)$.

$$f(x) = \ln(5x^2 - 4)$$

$$f'(x) = \frac{1}{5x^2 - 4} \cdot 10x$$

$$f'(x) = \frac{10x}{5x^2 - 4}$$

$$f''(x) = \frac{10(5x^2 - 4) - 10x(10x)}{(5x^2 - 4)^2}$$

$$f''(x) = \frac{50x^2 - 40 - 100x^2}{(5x^2 - 4)^2}$$

$$f''(x) = \frac{-50x^2 - 40}{(5x^2 - 4)^2}$$

10. Compute the derivative of $h(x) = e^{\log(x^2+1)}$ at $x = 3$.

$$h(x) = e^{\log(x^2+1)}$$

$$h'(x) = e^{\log(x^2+1)} \cdot \frac{1}{(x^2+1)\ln(10)} \cdot 2x$$

$$h'(3) = e^{\log((3)^2+1)} \cdot \frac{2(3)}{((3)^2+1)\ln(10)}$$

$$h'(3) = e^{\log(10)} \cdot \frac{6}{10\ln(10)}$$

Note:
$\log(10) = 1$

$$= e^1 \cdot \frac{3}{5\ln(10)}$$

$$h'(3) = \frac{3e}{5\ln(10)}$$

11. Compute the second derivative $y = \left(\sin^{-1}(2x)\right)^2$.

$$y = (\sin^{-1}(2x))^2$$

$$\frac{dy}{dx} = 2(\sin^{-1}(2x))^1 \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$\frac{dy}{dx} = \frac{4\sin^{-1}(2x)}{\sqrt{1-4x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{4\left(\frac{1}{\sqrt{1-(2x)^2}}\right)\cdot 2 \cdot \left(\sqrt{1-4x^2}\right) - \frac{1}{2}(1-4x^2)^{-1/2}(-8x)(4\sin^{-1}(2x))}{\left(\sqrt{1-4x^2}\right)^2}$$

$$\frac{d^2y}{dx^2} = \left(\frac{8 + 16x\sin^{-1}(2x)(1-4x^2)^{-1/2}}{(1-4x^2)}\right)\cdot\frac{(1-4x^2)^{1/2}}{(1-4x^2)^{1/2}}$$

$$\frac{d^2y}{dx^2} = \frac{8\sqrt{1-4x^2} + 16x\sin^{-1}(2x)}{(1-4x^2)^{3/2}}$$

12. Write the equation of the line tangent to the curve $N(t) = \dfrac{10t}{1+\frac{t}{10}}$ at $t = 2$.

$$N(2) = \frac{10(2)}{1+\frac{2}{10}} = \frac{20}{\frac{6}{5}} = \frac{100}{6} = \frac{50}{3}$$

$$N(t) = \frac{10t}{1+\frac{1}{10}t}$$

$$N'(t) = \frac{10(1+\frac{1}{10}t) - \frac{1}{10}(10t)}{(1+\frac{1}{10}t)^2}$$

$$N'(t) = \frac{10+t-t}{(1+\frac{1}{10}t)^2} = \frac{10}{(1+\frac{1}{10}t)^2}$$

$$N'(2) = \frac{10}{(\frac{6}{5})^2} = \frac{10}{\frac{36}{25}} = \frac{250}{36} = \frac{125}{18}$$

Equation of Tangent Line:

$$y - \frac{50}{3} = \frac{125}{18}(x-2)$$

or

$$y = \frac{125}{18}(x-2) + \frac{50}{3}$$

13. Compute the derivative, $\frac{dy}{dx}$ of $4x = \frac{3+y^3}{y^2+x}$.

$$4x = \frac{3+y^3}{y^2+x}$$

$$4xy^2 + 4x^2 = 3 + y^3$$

$$4y^2 + 8y\frac{dy}{dx}\cdot x + 8x = 0 + 3y^2\frac{dy}{dx}$$

$$4y^2 + 8x = 3y^2\frac{dy}{dx} - 8xy\frac{dy}{dx}$$

$$4y^2 + 8x = \frac{dy}{dx}(3y^2 - 8xy)$$

$$\frac{4y^2+8x}{3y^2-8xy} = \frac{dy}{dx}$$

14. Determine the unit tangent vector to the curve $\mathbf{r}(t) = (t^2)\mathbf{i} + (3t^3)\mathbf{j}$ at the point where $t = -1$.

$$\vec{r}(t) = (t^2)\vec{i} + 3t^3\vec{j}$$

$$\vec{r}\,'(t) = 2t\,\vec{i} + 9t^2\vec{j} = \langle 2t, 9t^2\rangle$$

$$\vec{r}\,'(-1) = \langle 2(-1), 9(-1)^2\rangle$$

$$\vec{r}\,'(-1) = \langle -2, 9\rangle$$

$$|\vec{r}\,'(-1)| = \sqrt{(-2)^2 + (9)^2}$$

$$= \sqrt{4+81}$$

$$= \sqrt{85}$$

$$\frac{\vec{r}\,'(-1)}{|\vec{r}\,'(-1)|} = \left\langle \frac{-2}{\sqrt{85}}, \frac{9}{\sqrt{85}} \right\rangle$$

15. Assume that $\mathbf{r}(t)$ is a position function for an object. Find the velocity vector(s) and the speed at the point $(3, 0)$ when $\mathbf{r}(t) = \langle t^2 - 6t + 8, t^4 - 26t^2 + 25 \rangle$.

$x = 3$
$y = 0$

$t^2 - 6t + 8 = 3$
$t^2 - 6t + 5 = 0$
$(t-5)(t-1) = 0$
$t = 5, \ t = 1$

$t^4 - 26t^2 + 25 = 0$
$(t^2 - 1)(t^2 - 25) = 0$
$t^2 - 1 = 0$ or $t^2 - 25 = 0$
$t = \pm 1$ or $t = \pm 5$

Two values of t
$t = 1, 5$

$\vec{r}\,'(t) = \langle 2t - 6, 4t^3 - 52t \rangle$

At $t = 1$
$\vec{r}\,'(1) = \langle 2(1) - 6, 4(1)^3 - 52(1) \rangle$
$= \langle -4, -48 \rangle$
$|\vec{r}\,'(1)| = \sqrt{(-4)^2 + (-48)^2} = \sqrt{2320}$

At $t = 5$
$\vec{r}\,'(5) = \langle 2(5) - 6, 4(5)^3 - 52(5) \rangle$
$= \langle 4, 240 \rangle$
$|\vec{r}\,'(5)| = \sqrt{(4)^2 + (240)^2} = \sqrt{57616}$

16. Determine the points on the curve, defined by the parametric equations $x(t) = e^{t^2+4t}$ and $y(t) = 5^{3t+2}$, where the tangent lines are horizontal and where they are vertical.

$x(t) = e^{t^2+4t}$
$\frac{dx}{dt} = e^{t^2+4t}(2t+4)$

Vertical Tangents $\frac{dx}{dt} = 0$
$0 = e^{t^2+4t}(2t+4)$
$e^{t^2+4t} \neq 0$ or $2t + 4 = 0$
$2t = -4$
$t = -2$

$x(-2) = e^{(-2)^2 + 4(-2)}$
$y(-2) = 5^{3(-2)+2}$

$(x, y) = (e^{-4}, 5^{-4})$

$y(t) = 5^{3t+2}$
$\frac{dy}{dt} = 5^{3t+2} \ln(5) \cdot (3)$

Horizontal Tangent $\frac{dy}{dt} = 0$
$0 = 5^{3t+2} \cdot \ln(5) \cdot 3$
$0 \neq 5^{3t+2}$

No Horizontal Tangents

17. An object is moving in a straight line. The position of the object is given by the equation $s(t) = 4t^3 - 9t^2 + 6t + 2$, where t is measured in seconds and s is measured in meters.

a. Compute the velocity and acceleration of the object at time t.

$s(t) = 4t^3 - 9t^2 + 6t + 2$

$v(t) = 12t^2 - 18t + 6$

$a(t) = 24t - 18$

b. When is the object at rest?

$v(t) = 0$

$12t^2 - 18t + 6 = 0$

$2t^2 - 3t + 1 = 0$

$(2t-1)(t-1) = 0$

$2t - 1 = 0$ or $t - 1 = 0$

$t = \frac{1}{2}$ sec or $t = 1$ sec

18. Suppose that the population size of a bacteria at time t days is $N(t) = N_0 7^{0.05t}$, $t \geq 0$ where $N_0 = 1200$ is the initial population. Compute the rate of growth of the bacteria population after 6 days.

$N'(t)$

$N(t) = N_0 \cdot 7^{0.05t}$

$N(t) = 1200 \cdot 7^{0.05t}$

$N'(t) = 1200 \cdot 7^{0.05t} \cdot \ln(7)(0.05)$

$N'(t) = 60\ln(7) \cdot 7^{0.05t}$

$N'(6) = 60\ln(7) \cdot 7^{0.05(6)}$

$N'(6) = 60\ln(7) \cdot 7^{0.3}$ bacteria/day

19. When an energy drink is removed from the refrigerator, its temperature is 35°F. After 30 minutes, in a room registering 72°F, the energy drink temperature has increased to 60°F.

a. What is the temperature of the drink after 45 minutes?

$T_a = 72°F$
$T_o = 35°F$

$T(t) = T_a + (T_o - T_a)e^{-kt}$

$T(t) = 72 + (35 - 72)e^{-kt}$

$T(t) = 72 - 37e^{-kt}$

$60 = 72 - 37e^{-k(1/2)}$

$-12 = -37e^{-1/2 k}$

$\frac{12}{37} = e^{-1/2 k}$

$\ln\left(\frac{12}{37}\right) = -\frac{1}{2}k$

$-2\ln\left(\frac{12}{37}\right) = k$

$T(t) = 72 - 37e^{2\ln(12/37)t}$

$T(3/4) = 72 - 37e^{2\ln(12/37)\cdot 3/4}$

$T(3/4) = 72 - 37e^{3/2\ln(12/37)} \approx 65.166°F$

b. At what time does the energy drink reach a temperature of 70°F?

$T(t) = 72 - 37e^{2\ln(12/37)t}$

$70 = 72 - 37e^{2\ln(12/37)t}$

$-2 = -37e^{2\ln(12/37)t}$

$\frac{2}{37} = e^{2\ln(12/37)t}$

$\ln\left(\frac{2}{37}\right) = 2\ln\left(\frac{12}{37}\right)t$

$\frac{\ln(2/37)}{2\ln(12/37)} = t$

$1.296 \text{ hours} \approx t$

$77.837 \text{ min} \approx t$

20. Compute the 26th derivative of $g(x) = x\sin(x)$.

$g(x) = x \cdot \sin(x)$

$\frac{dg(x)}{dx} = 1\sin(x) + \cos(x)\cdot x$

$\frac{d^2g(x)}{dx^2} = \cos(x) + (-\sin(x))x + 1\cos(x)$
$= 2\cos(x) - x\sin(x)$

$\frac{d^3g(x)}{dx^3} = -2\sin(x) - 1\sin(x) - \cos(x)\cdot x$
$= -3\sin(x) - x\cos(x)$

$\frac{d^4g(x)}{dx^4} = -3\cos(x) - 1\cos(x) + \sin(x)(x)$
$= -4\cos(x) + x\sin(x)$

$\frac{d^5g(x)}{dx^5} = 4\sin(x) + 1\sin(x) + \cos(x)\cdot x$
$= 5\sin(x) + x\cos(x)$ ← back to 1st derivative with coefficient change

$\vdots$

$\frac{d^{26}g(x)}{dx^{26}} = 26\cos(x) - x\sin(x)$ like second derivative

21. Compute the third derivative of $h(x) = e^{-x^2+1}$.

$h(x) = e^{-x^2+1}$

$h'(x) = e^{-x^2+1}(-2x)$
$= -2x\cdot e^{-x^2+1}$

$h''(x) = -2e^{-x^2+1} + e^{-x^2+1}\cdot(-2x)(-2x)$
$= -2e^{-x^2+1}(1-2x^2)$

$h'''(x) = -2e^{-x^2+1}(-2x)(1-2x^2) + (-4x)(-2e^{-x^2+1})$
$= -2e^{-x^2+1}[-2x+4x^3-4x]$
$= -2e^{-x^2+1}[4x^3-6x]$