Math 308: Week-in-Review 1 (1.1, 1.2, 1.3 & 2.1)

1. Verify that each of the given functions is a solution of the corresponding differential equation.

   (a) $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 12y = 0, \qquad y(t) = e^{4t},\ y(t) = e^{-3t}$

   Plug in

   $y_1(t) = e^{4t}$

   $y_1'(t) = 4e^{4t}$

   $y_1''(t) = 16e^{4t}$

   $y_1''(t) - y_1'(t) - 12y_1(t)$

   $= 16e^{4t} - 4e^{-4t} - 12e^{4t}$

   $= 0$ ✓

   $y_2(t) = e^{-3t}$

   $y_2'(t) = -3e^{-3t}$

   $y_2''(t) = 9e^{-3t}$

   $y_2''(t) - y_2'(t) - 12y(t)$

   $= 9e^{-3t} - (-3e^{-3t}) - 12e^{-3t}$

   $= 12e^{-3t} - 12e^{-3t}$

   $= 0$ ✓

   (b) $\dfrac{dy}{dt} + y = \sin(t), \qquad y(t) = Ae^{-t} - \dfrac{1}{2}\cos(t) + \dfrac{1}{2}\sin(t).$

   Plug in

   $y'(t) = -Ae^{-t} + \frac{1}{2}\sin(t) + \frac{1}{2}\cos(t)$

   $y'(t) + y(t) = -\cancel{A}e^{-t} + \frac{1}{2}\sin(t) + \cancel{\frac{1}{2}\cos(t)} + \cancel{A}e^{-t} - \cancel{\frac{1}{2}\cos(t)} + \frac{1}{2}\sin(t)$

   $= \sin(t)$ ✓

(c) $y''' - 2y'' = 0$, $y(x) = A + Bx + Ce^{2x}$ (where A, B and C are constants).

$$\left.\begin{aligned} y'(x) &= B + 2Ce^{2x} \\ y''(x) &= 4Ce^{2x} \\ y'''(x) &= 8Ce^{2x} \end{aligned}\right\}$$

$$\begin{aligned} y'''(x) - 2y''(x) &= 8Ce^{2x} - 2\cdot 4Ce^{2x} \\ &= 8Ce^{2x} - 8Ce^{2x} \\ &= 0 \ \checkmark \end{aligned}$$

2. (a) Find values of A and B so that the function $y(x) = Ae^{4x} + Be^{-3x}$ is a solution of the initial value problem $y'' - y' - 12y = 0$, $y(0) = 1$, $y'(0) = -1$.

↳ check 1(a) that y(x) is a soln

$$y(0) = Ae^{4\cdot 0} + Be^{-3\cdot 0} = A + B = 1$$

$$y'(x) = 4Ae^{4x} - 3Be^{-3x} \Rightarrow y'(0) = 4Ae^{4\cdot 0} - 3Be^{-3\cdot 0} = 4A - 3B = -1$$

Solve $\begin{cases} A + B = 1 & * \\ 4A - 3B = -1 & ** \end{cases}$

$$\boxed{y(x) = \frac{2}{7}e^{4x} + \frac{5}{7}e^{-3x}}$$

$*$ $A = 1 - B$

$**$ $4(1-B) - 3B = -1$

$4 - 4B - 3B = -1$

$4 - 7B = -1 \Rightarrow 5 = 7B \Rightarrow B = 5/7$

$*$ $A = 1 - 5/7 = 2/7$

(b) Find the values of A, B and C so that $y(x) = A + Bx + Ce^{2x}$ is a solution of the initial value problem $y''' - 2y'' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 3$.

$$y(0) = A + B\cdot 0 + Ce^{2\cdot 0} = A + C = 0 \Rightarrow A = -C \quad *$$

$$y'(x) = B + 2Ce^{2x} \Rightarrow y'(0) = B + 2Ce^{2\cdot 0} = B + 2C = 1 \quad **$$

$$y''(x) = 4Ce^{2x} \Rightarrow y''(0) = 4Ce^{2\cdot 0} = 4C = 3 \Rightarrow C = 3/4$$

$*$ $A = -C \Rightarrow A = -3/4$

$**$ $B = 1 - 2C \Rightarrow B = 1 - 2\cdot 3/4 = 1 - \frac{3}{2} = -\frac{1}{2}$

$$\boxed{y(x) = -\frac{3}{4} - \frac{1}{2}x + \frac{3}{4}e^{2x}}$$

3. Use direct integration to find the general solution of each of the differential equations

$$\frac{dy}{dx} = f(x) \Rightarrow y(x) = \int f(x)\,dx$$

(a) $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-16}}$

substitution

$$\begin{cases} u = x^2 - 16 \\ du = 2x\,dx \end{cases}$$

$$y(x) = \int \frac{x}{\sqrt{x^2-16}}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-16}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du = 2\cdot\frac{1}{2}\cdot u^{1/2} + C = \sqrt{u} + C$$

$$\boxed{y(x) = \sqrt{x^2-16} + C}$$

(b) $\dfrac{dy}{dx} = xe^{-x}$

integration by parts

$$y(x) = \int x e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + C$$

+	x	e^{-x}
−	1	$-e^{-x}$
+	0	e^{-x}

$$\int x e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

$$\boxed{y(x) = -xe^{-x} - e^{-x} + C}$$

(c) $\dfrac{dy}{dx} = \dfrac{1}{x^2-16} = \dfrac{1}{(x+4)(x-4)}$

partial fractions

$$\frac{1}{(x+4)(x-4)} = \frac{A}{x+4} + \frac{B}{x-4} = \frac{-1}{8(x+4)} + \frac{1}{8(x-4)}$$

$$1 = A(x-4) + B(x+4)$$

$x = 4:\quad 1 = A\cdot 0 + 8B \Rightarrow B = 1/8$

$x = -4:\quad 1 = -8A + 0\cdot B \Rightarrow A = -1/8$

$$y(x) = \int \frac{1}{x^2-16}\,dx$$

$$= \int \frac{1}{(x+4)(x-4)}\,dx$$

$$= -\frac{1}{8}\int \frac{1}{x+4}\,dx + \frac{1}{8}\int \frac{1}{x-4}\,dx$$

$$\boxed{y(x) = -\frac{1}{8}\ln|x+4| + \frac{1}{8}\ln|x-4| + C}$$

4. For each of the following, determine whether it is an ODE or PDE. Additionally, state (a) the order of the differential equation and (b) whether it is linear or nonlinear.

   (a) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x^3$ ODE, 2nd order linear

   (b) $\underbrace{y\dfrac{dy}{dx} + y^4}_{\text{nonlinear}} = \sin(x)$ ODE, 1st order, nonlinear

   (c) $\dfrac{1}{c^2}\dfrac{\partial^2 z}{\partial t^2} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$, c is constant PDE, 2nd order, linear

   (d) $\left(\dfrac{dy}{dx}\right)^2 + y = 0$ ODE, 1st order, nonlinear

   nonlinear

   (e) $u_t + uu_x = \sigma u_{xx}$, σ is a constant PDE, 2nd order, nonlinear

   nonlinear

   (f) $r^2R''(r) + rR'(r) + (r^2 - \alpha^2)R(r) = 0$ α is a constant ODE, 2nd order, linear

   (g) $u_x + u_y + u^2 = 0$ PDE, 1st order, nonlinear

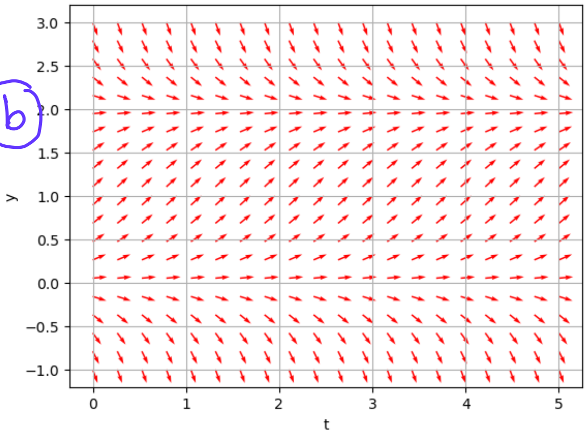
   nonlinear

   (h) $t^4 + w^{(7)}(t) + \tan(t)w''(t) = \sin(t)$ ODE, 7th order, linear

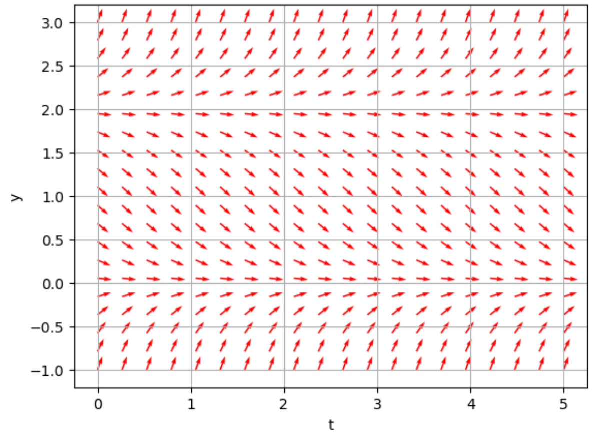
5. Consider the following list of differential equations, some of which produced the direction fields shown in the figures. Identify the differential equation that corresponds to the given direction field.

(a) $y' = y(y-2)^2$ (b) $y' = y(2-y)$ (c) $y' = y+2$ (d) $y' = -y(2-y)$ (e) $y' = -2-y$

b



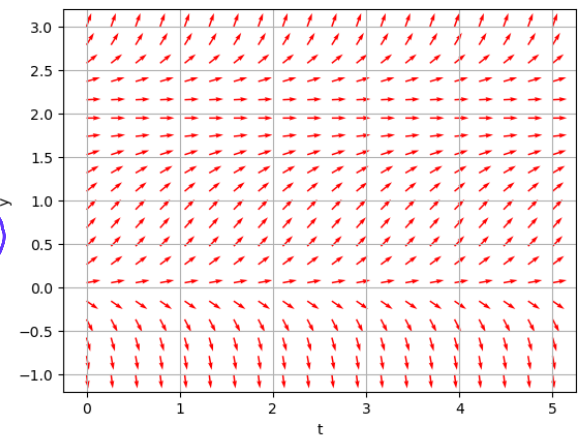
(a)



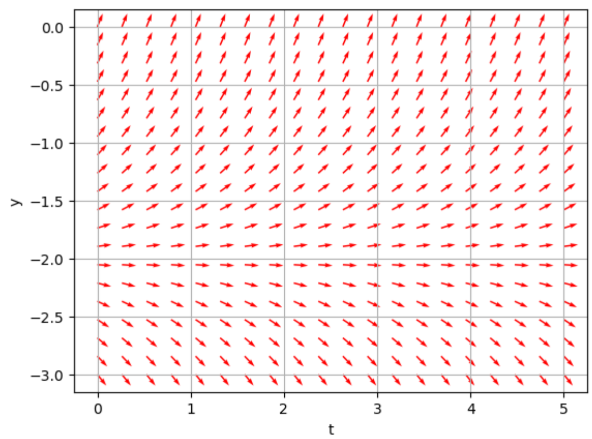
d

(b)

a



(c)



c

(d)

Figure 1: Direction fields.

6. Use the direction fields below to determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

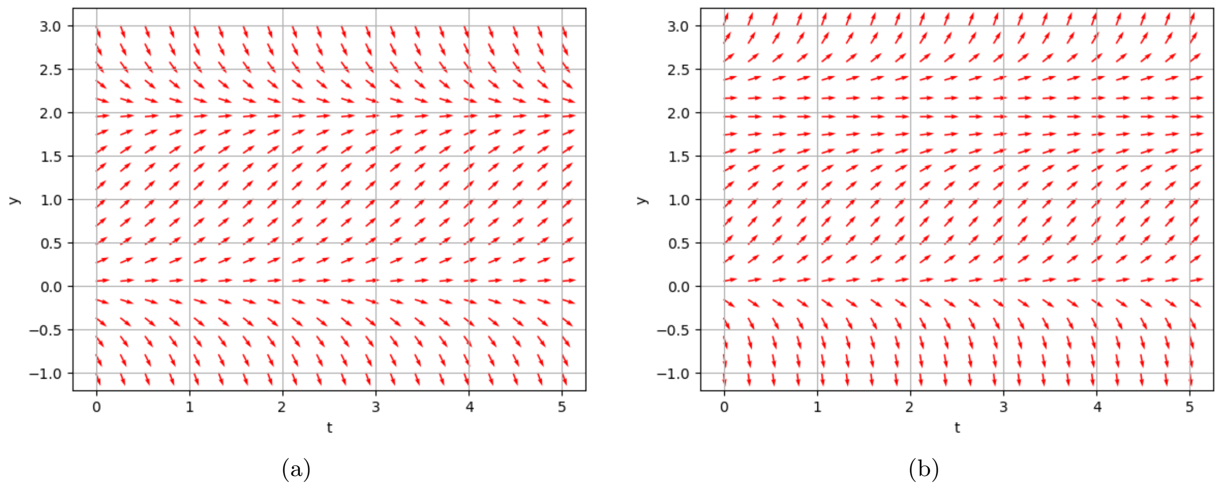


Figure 2: Direction fields.

ⓐ If $0 < y(0) < \infty$

$\lim_{t \to \infty} y(t) = 2$

If $y(0) = 0$

$\lim_{t \to \infty} y(t) = 0$

If $y(0) < 0$

$\lim_{t \to \infty} y(t) = -\infty$

ⓑ If $y(0) > 2$

$\lim_{t \to \infty} y(t) = \infty$

If $y(0) = 2$

$\lim_{t \to \infty} y(t) = 2$

If $0 < y(0) < 2$

$\lim_{t \to \infty} y(t) = 2$

If $y(0) = 0$

$\lim_{t \to \infty} y(t) = 0$

If $y(0) < 0$

$\lim_{t \to \infty} y(t) = -\infty$

7. Solve each of the following differential equations.

(a) $ty' + y = t^2$

$\frac{d}{dt}[ty] = t^2$

* product rule! *

In general, if $Q(t)y' + R(t)y = G(t)$ and $Q'(t) = R(t)$ then $\frac{d}{dt}\left[Q(t)y\right] = G(t)$

* product rule *

$$\frac{d}{dt}[ty] = t^2$$

$$ty = \int t^2\,dt = \frac{t^3}{3} + C$$

$$y(t) = \frac{1}{t}\left[\frac{t^3}{3} + C\right] \Rightarrow \boxed{y(t) = \frac{t^2}{3} + \frac{C}{t}}$$

* $\frac{C}{t}$ is a function of t & cannot be replaced by a constant *

(b) $(e^t + 1)\dfrac{dy}{dt} + e^t y = t, \quad y(0) = -1.$

$$\frac{d}{dt}\left[(e^t+1)y\right] = t$$

$$(e^t+1)y = \int t\,dt$$

$$(e^t+1)y = \frac{t^2}{2} + C$$

$$y = \frac{1}{e^t+1}\left(\frac{t^2}{2} + C\right)$$

$$y(t) = \frac{t^2}{2(e^t+1)} + \frac{C}{e^t+1}$$

$$y(0) = \frac{C}{2} = -1 \Rightarrow C = -2$$

$$\boxed{y(t) = \frac{t^2}{2(e^t+1)} - \frac{2}{e^t+1}}$$

(c) $\underbrace{\frac{dy}{dt} + \frac{2t}{1+t^2}y}_{\text{* need integrating factor } \mu(t) \text{ *}} = \frac{1}{1+t^2}, \quad y(0) = 4.$ } standard form $y' + p(t)y = g(t)$

$p(t) = \frac{2t}{1+t^2}$, $g(t) = \frac{1}{1+t^2}$

$$\mu(t) = e^{\int p(t)dt}$$
$$= e^{\int \frac{2t}{1+t^2}dt}$$
$u = 1+t^2$, $du = 2t\,dt$
$$= e^{\int \frac{1}{u}dt} = e^{\ln u} = u = 1+t^2$$

$$\mu(t)\frac{dy}{dt} + \mu(t)\frac{2t}{1+t^2}y = \mu(t)\frac{1}{1+t^2} \Rightarrow \underbrace{(1+t^2)\frac{dy}{dt} + 2ty}_{\frac{d}{dt}\left[(1+t^2)y\right]} = 1$$

$$\frac{d}{dt}\left[(1+t^2)y\right] = 1$$

$$(1+t^2)y = \int 1 \cdot dt = t + C$$

$$y = \frac{t}{1+t^2} + \frac{C}{1+t^2} \Rightarrow \boxed{y(t) = \frac{t}{1+t^2} + \frac{4}{1+t^2}}$$

IC: $y(0) = C = 4$

(d) $\underbrace{t\frac{dy}{dt} + (t+1)y}_{\text{* need integrating factor } \mu(t)} = t, \quad y(\ln\ 2) = 1,\ t > 0.$ → $\frac{dy}{dt} + (1 + 1/t)y = 1$ * standard form *

$p(t) = 1 + \frac{1}{t}$, $g(t) = 1$

$$\mu(t) = e^{\int(1+1/t)dt} = e^{t + \ln t} = e^t \cdot e^{\ln t} = te^t \Rightarrow \mu(t)\frac{dy}{dt} + \mu(t)\left(1+\frac{1}{t}\right)y = \mu(t)$$

$$\underbrace{te^t\frac{dy}{dt} + (te^t + e^t)y}_{\frac{d}{dt}\left[te^t y\right]} = te^t$$

$$\frac{d}{dt}\left[te^t y\right] = te^t$$

integration by parts!

$$te^t y = \int te^t\,dt$$

+	t	te^t
−	1	te^t
+	0	e^t

$$= te^t - e^t + C$$

$$y(t) = 1 - \frac{1}{t} + \frac{C}{te^t} \Rightarrow \boxed{y(t) = 1 - \frac{1}{t} + \frac{2}{te^t}} \text{ or } \boxed{y(t) = 1 - \frac{1}{t} + \frac{2}{t}e^{-t}}$$

$$y(\ln 2) = 1 - \frac{1}{\ln 2} - \frac{C}{2\ln 2} = 1 \Rightarrow \frac{C}{2\ln 2} = \frac{1}{\ln 2} \Rightarrow C = 2$$

(e) Find the solution of the initial value problem and describe its behavior for large t.

$$y' + \frac{1}{4}y = 3 + 2\cos(2t), \quad y(0) = 0$$

* standard form *

$p(t) = \frac{1}{4}$, $g(t) = 3 + 2\cos(2t)$

$\mu(t) = e^{\int p(t)dt} = e^{t/4}$

* multiply by $\mu(t)$ *

$\mu(t)y' + \mu(t)\frac{1}{4}y = \mu(t)\cdot 3 + \mu(t)\cdot 2\cos(2t)$

$\underbrace{e^{t/4}y' + \frac{1}{4}e^{t/4}y}_{} = 3e^{t/4} + 2e^{t/4}\cos(2t)$

$\frac{d}{dt}\left[e^{t/4}y\right] = 3e^{t/4} + 2e^{t/4}\cos(2t)$

$e^{t/4}y = 3\int e^{t/4}dt + 2\underbrace{\int e^{t/4}\cos(2t)\,dt}_{\text{* by parts *}}$

$3\int e^{t/4}dt = 12e^{t/4}$

$+$	$\cos(2t)$	$e^{t/4}$
$-$	$-2\sin(2t)$	$4e^{t/4}$
$+$	$-4\cos(2t)$	$16e^{t/4}$

$\int e^{t/4}\cos(2t)dt = 4e^{t/4}\cos(2t) + 32e^{t/4}\sin(2t) - 64\int e^{t/4}\cos(2t)dt$

$\Downarrow$

$65\int e^{t/4}\cos(2t) = 4e^{t/4}\cos(2t) + 32e^{t/4}\sin(2t)$

$\int e^{t/4}\cos(2t) = \frac{4}{65}e^{t/4}\cos(2t) + \frac{32}{65}e^{t/4}\sin(2t)$

$= 12e^{t/4} + 2\left[\frac{4}{65}e^{t/4}\cos(2t) + \frac{32}{65}e^{t/4}\sin(2t)\right] + C$

$y = 12 + \frac{8}{65}\cos(2t) + \frac{32}{65}\sin(2t) + Ce^{-t/4}$

$y(0) = 12 + \frac{8}{65} + C = 0 \Rightarrow C = -12 - \frac{8}{65} = \frac{-788}{65}$

$$\boxed{y(t) = 12 + \frac{8}{65}\cos(2t) + \frac{32}{65}\sin(2t) - \frac{788}{65}e^{-t/4}}$$

this term vanishes as $t \to \infty$

$y(t) \sim 12 + \frac{8}{65}\cos(2t) + \frac{32}{65}\sin(2t)$ for large t

The steady state is an oscillation around $y = 12$.