Example 1. *True/False.*

(a) *The curve* $\mathbf{r}(t) = \langle t^2, 2t+1, t \rangle$ *lies on the plane* $y - 2z = 1$.

True   *False*

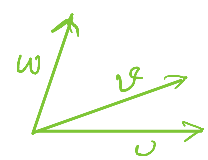
$x = t^2, \ \underbrace{y = 2t+1, \ z = t}_{y = 2z+1}$

(b) *The curvature of a straight line is zero.*

True   *False*

(c) *Let* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *be orthogonal to each other. Then no planes can contain all three vectors.*

True   *False*



(d) *The line* $x = 1+2t, \ y = 1+t, \ z = 3-3t$ *is orthogonal to the plane* $2x + y + 2z = 1$.

True   *False*

Direction vector : $\langle 2, 1, -3 \rangle$

A normal : $\langle 2, 1, 2 \rangle$

$\langle 2, 1, -3 \rangle$ is NOT parallel to $\langle 2, 1, 2 \rangle$

(e) *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two nonzero vectors. Then the vectors* $\text{proj}_{\mathbf{a}}\mathbf{b}$ *and* $\mathbf{a}$ *are parallel.*

True   *False*

$$\text{proj}_a b = \underbrace{\left(\frac{a \cdot b}{|a|}\right)}_{\text{comp}_a b} \frac{a}{|a|} = \left(\frac{a \cdot b}{|a|^2}\right) a$$

Example 2 (12.1). *Find the intersection of the sphere $x^2+y^2+z^2-4x+6y-10z+29=0$ with*

(a) *the xy-plane.* $x^2-4x+\underline{4}+y^2+6y+\underline{9}+z^2-10z+\underline{25}+29=38$

$$\boxed{(x-2)^2+(y+3)^2+(z-5)^2=9}$$

xy-plane : $z=0 \Rightarrow (x-2)^2+(y+3)^2+(0-5)^2=9$

$(x-2)^2+(y+3)^2=-16$, impossible.

The sphere doesn't touch the xy-plane.

(b) *the plane $z=8$.*

$z=8 \Rightarrow (x-2)^2+(y+3)^2+(8-5)^2=9$

$(x-2)^2+(y+3)^2=0$

$\Rightarrow$ The sphere touches the plane $z=8$ only at the point $(2,-3,8)$.

Example 3 (12.2). *Consider two vectors $\mathbf{v}=\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $\mathbf{w}=2\mathbf{i}+\mathbf{k}$. If a vector $\mathbf{a}$ is in the direction of $\mathbf{v}-\mathbf{w}$ and has magnitude 3 units, find the components of the vector $\mathbf{a}$.*

$v-w=\langle 1,-3,2\rangle-\langle 2,0,1\rangle=\langle -1,-3,1\rangle$

$|v-w|=\sqrt{11}$

Unit vector in the direction of $v-w=\frac{1}{\sqrt{11}}\langle -1,-3,1\rangle$.

Desired vector $=\frac{3}{\sqrt{11}}\langle -1,-3,1\rangle$

Example 4 (12.3). *Find the work done by a force* $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ *that moves an object from the point* $A(1, 0, 2)$ *along a straight line to the point* $B(2, 4, 3)$. *Also, find the angle between the displacement and force vectors.*

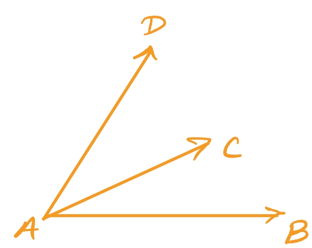
Force $F = \langle 3, 2, -5 \rangle$, Displacement $= \langle 2-1, 4-0, 3-2 \rangle$
$= \langle 1, 4, 1 \rangle$

Work $W = F \cdot D = 3 + 2(4) + (-5)(1) = 6$

$$\cos\theta = \frac{F \cdot D}{|F|\,|D|} = \frac{6}{\sqrt{38}\sqrt{18}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{6}{\sqrt{38}\sqrt{18}}\right)$$

Example 5 (12.4). *Determine whether or not the points* $A(0, 0, 1)$, $B(1, 2, 1)$, $C(1, 0, -1)$, *and* $D(3, 2, 1)$ *lie in the same plane.*



The points lie on a plane if the volume of the parallelepiped formed by $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ is zero.

And recall that the volume of the parallelpiped $= |AB \cdot (\vec{AC} \times \vec{AD})|$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & -2 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$
$$= 4 - 2(6) + 0 = -8$$

Volume $= |-8| = 8 \neq 0$.

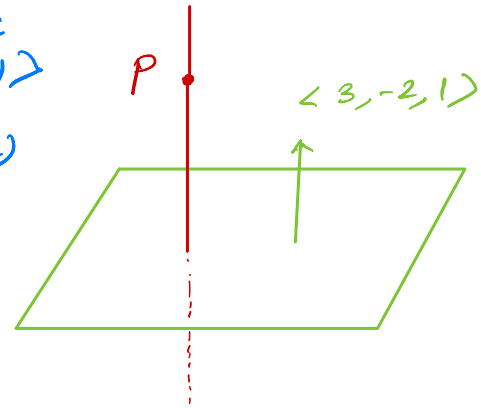
So, the points do NOT lie in the same plane.

Example 6 (12.5). *Suppose a line L_1 passes through the point $P(1, 4, -2)$ and is orthogonal to the plane $3x - 2y + z = 35$.*

(a) Determine symmetric equations of the line L_1.

A direction vector to L_1 is $v = \langle \overset{a}{3}, \overset{b}{-2}, \overset{c}{1} \rangle$

A point on L_1 : $(x_0, y_0, z_0) = (1, 4 - 2)$



Symmetric equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\frac{x - 1}{3} = \frac{y - 4}{-2} = z + 2$$

(b) Find the point of intersection of the line L_1 and the plane.

Substitute $x = 1 + 3t$, $y = 4 - 2t$, $z = -2 + t$ into $3x - 2y + z = 35$

$$3 + 9t - 8 + 4t - 2 + t = 35$$

$$14t = 42 \Rightarrow \boxed{t = 3}$$

Point of intersection:

$$(x, y, z) = \langle 10, -2, 1)$$

Example 7 (12.5). *Determine whether the lines L_1 and L_2 are parallel, intersecting, or skew.*

Direction vectors:

$\langle 3, -2, 1\rangle \leftarrow$ $L_1: \dfrac{x+2}{3} = \dfrac{y-4}{-2} = z = t$

$\langle 2, 3, -2\rangle \leftarrow$ $L_2: \dfrac{x+1}{2} = \dfrac{y+3}{3} = \dfrac{z-1}{-2}. = s$

L_1: $x = -2+3t$, $y = 4-2t$, $z = t$

L_2: $x = -1+2s$, $y = -3+3s$, $z = 1-2s$

Since the direction vectors are not parallel, the lines are NOT parallel.

Testing intersection:

$-2+3t = -1+2s$ $\quad$ $4-2t = -3+3s$, $\quad t = 1-2s$

$4-2(1-2s) = -3+3s$

$4-2+4s = -3+3s$

$\boxed{s = -5}$

$\boxed{t = 11}$

substituting $s=-5$, $t=11$ into $-2+3t = -1+2s$:

$-2+3(11) = -1+2(-5)$

$31 = -11$, a contradiction.

This means the lines do NOT intersect, either.

Hence, the lines are skew.

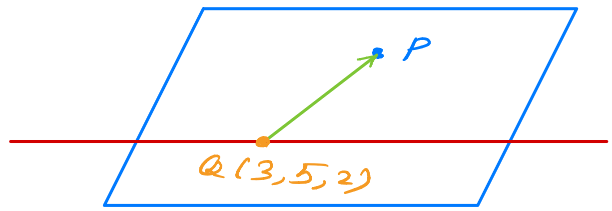
Example 8 (12.5). *Find an equation of the plane that passes through the point* $P(1, -2, 3)$ *and contains the line* $x = 3 + t, y = 5, z = 2 - 5t$.

x_0, y_0, z_0

$t = 0 \Rightarrow (x, y, z) = (3, 5, 2)$ is a point on the line.

A direction vector to the line is

$v = \langle 1, 0, -5 \rangle$.



So, a normal to the plane is $\vec{n} = v \times \overrightarrow{PQ}$

$$= \langle 1, 0, -5 \rangle \times \langle 2, 7, -1 \rangle$$

$$= \begin{vmatrix} i & j & k \\ 1 & 0 & -5 \\ 2 & 7 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -5 \\ 7 & -1 \end{vmatrix} i - \begin{vmatrix} 1 & -5 \\ 2 & -1 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} k$$

$$= 35i - 9j + 7k = \langle 35, -9, 7 \rangle$$

An equation of the plane is

$$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\langle 35, -9, 7 \rangle \cdot \langle x - 1, y + 2, z - 3 \rangle = 0$$

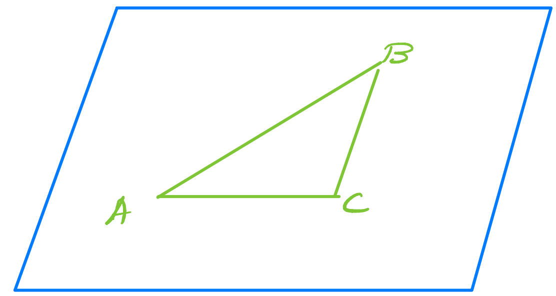
$$35x - 9y + 7z - 35 - 18 - 21 = 0$$

$$35x - 9y + 7z = 74$$

Example 9 (12.5). *Consider that a plane P contains the triangle ABC with vertices $A(1,2,2)$, $B(1,3,3)$, and $C(3,1,0)$.*

x_0, y_0, z_0

(a) Determine an equation of the plane P.



A normal to the plane is

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 2 & -1 & -2 \end{vmatrix} = -i + 2j - 2k = \langle -1, 2, -2 \rangle.$$

An equation of the plane is

$$\langle -1, 2, -2 \rangle \cdot (x-1, y-2, z-2) = 0$$

$$-x + 2y - 2z + 1 - 4 + 4 = 0$$

$$-x + 2y - 2z + 1 = 0$$

(b) Which of the following lines is orthogonal to the plane P?

Direction vector

$L_1: \quad x = 3 + t, \quad y = 4 + 2t, \quad z = 3 + 2t \rightarrow \langle 1, 2, 2 \rangle$

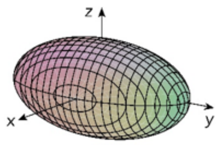
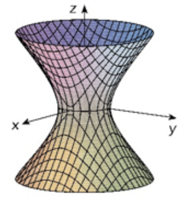
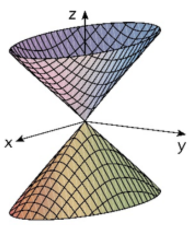
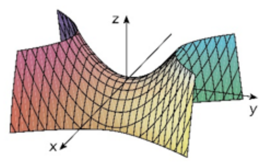
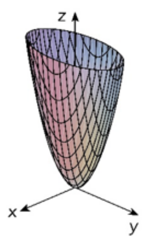
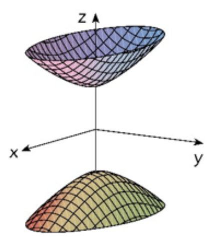
$L_2: \quad x = 1 - t, \quad y = 4 - 2t, \quad z = 3 - 2t \rightarrow \langle -1, -2, -2 \rangle$

$L_3: \quad x = -1 + 3t, \quad y = 2 + 2t, \quad z = -2 - 2t \rightarrow \langle 3, 2, -2 \rangle$

$L_4: \quad x = 1 - 3t, \quad y = 5 + 6t, \quad z = 7 - 6t \rightarrow \langle -3, 6, -6 \rangle$

None of the above lines.

As a direction vector of L_4 is parallel to a normal vector of the plane P, the line L_4 is orthogonal to the plane.

Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ "A bunch of ellipses stacked together" Special case: If $a=b=c$, we have a sphere	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas. *Whichever variable is negative corresponds to the axis of symmetry
Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas, except when $x=0$ or $y=0$, then the traces are pairs of lines	Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ In the xy plane, the traces are hyperbolas. In the xz or yz plane, the traces are parabolas.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are parabolas.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses if $z>c$ or $z<-c$ In the xz or yz planes, the traces are hyperbolas.

Example 10 (12.6). *Identify and sketch the following quadric surfaces.*

$$(y-2)^2-(x+1)^2-z^2+4y+2x-6=0$$

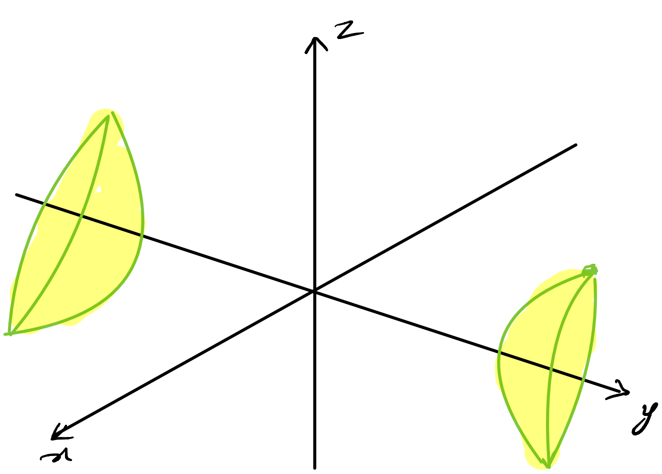
$$y^2-\cancel{4y}+4-x^2-\cancel{2x}-1-z^2+\cancel{4y}+\cancel{2x}-6=0$$

$$y^2-x^2-z^2-3=0$$

$$y^2=x^2+z^2+3$$

OR

$$y=\pm\sqrt{x^2+z^2+3}$$



Example 11 (13.1). *Find all the points of intersection of the curve* $\mathbf{r}(t) = \langle t, 2t, t^2+4 \rangle$ *and the plane* $x + 2y - z = 0$.

$r(t)$: $x = t,\ y = 2t,\ z = t^2 + 4$.

Substituting into $x + 2y - z = 0$;

$$t + 4t - t^2 - 4 = 0$$

$$-t^2 + 5t - 4 = 0 \Rightarrow t^2 - 5t + 4 = 0 \Rightarrow (t-4)(t-1) = 0$$

$$\Rightarrow t = 1 \text{ or } t = 4$$

$t = 1 \Rightarrow (x, y, z) = (1, 2, 5)$

$t = 4 \Rightarrow (x, y, z) = (4, 8, 20)$

Recall integration by parts: $\int u\,dv = uv - \int v\,du$

Example 12 (13.2). *Compute the integral* $\int_0^1 \mathbf{r}(t)$, *where* $\mathbf{r}(t) = te^{-t}\mathbf{i} + \dfrac{1}{t+1}\mathbf{j} + 3t^2\mathbf{k}$.

$$\int \underbrace{t}_{u}\ \underbrace{e^{-t}dt}_{dv} = -te^{-t} + \int e^{-t}dt \qquad \left(\begin{array}{l} u = t \Rightarrow du = dt \\ dv = e^{-t} \Rightarrow v = -e^{-t} \end{array}\right)$$

$$= -te^{-t} - e^{-t}$$

$$\int_0^1 te^{-t}dt = \left[-te^{-t} - e^{-t}\right]_0^1 = \left[(-e^{-1} - e^{-1}) - (0 - 1)\right]$$

$$= 1 - 2e^{-1} = \frac{e-2}{e}$$

$$\int_0^1 \frac{1}{t+1}dt = \ln(t+1)\Big|_0^1 = \ln(2) - \ln 1 = \ln 2$$

$$\int_0^1 3t^2 dt = t^3\Big|_0^1 = 1$$

So, $\displaystyle\int_0^1 \left(te^{-t}i + \frac{1}{t+1}j + 3t^2K\right)dt = \left(\frac{e-2}{e}\right)i + \ln 2\, j + K$

Example 13 (13.1). *Consider the vector function* $\mathbf{r}(t) = \left\langle e^{-6t}, \ln(2t+1), \dfrac{t^2-9}{t-3} \right\rangle$.

(a) *Find the domain of* $\mathbf{r}$.

e^{-6t} is defined for any $t \in \mathbb{R}$.

$\ln(2t+1)$ " " for $2t+1>0 \Rightarrow t > -\frac{1}{2}$

$\dfrac{t^2-9}{t-3}$ " " when $t \neq 3$.

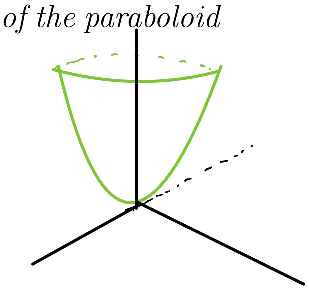
So, Domain $= (-\frac{1}{2}, 3) \cup (3, \infty)$.

(b) *Find* $\lim_{t\to 3} \mathbf{r}(t)$. $= \left\langle \lim_{t\to 3} e^{-6t}, \lim_{t\to 3} \ln(2t+1), \lim_{t\to 3} \dfrac{t^2-9}{t-3} \right\rangle$

$= \left\langle e^{-18}, \ln 7, \lim_{t\to 3} t+3 \right\rangle$

$= \left\langle e^{-18}, \ln 7, 6 \right\rangle$

Example 14 (13.1). *Find parametric equations for the curve of intersection of the paraboloid* $z = \frac{1}{2}(x^2+y^2)$ *and the plane* $z = x$.



$z = x \Rightarrow x = \frac{1}{2}(x^2+y^2)$

$2x = x^2 + y^2$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + (y-0)^2 = 1$.

$\Rightarrow$ The projection of the intersection onto xy-plane is the circle $(x-1)^2 + (y-0)^2 = 1$

$x = 1 + \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$.

$z = x = 1 + \cos t$.

That is, $x = 1+\cos t,\ y = \sin t,\ z = 1+\cos t,\ 0 \le t \le 2\pi$.

Example 15 (13.2)**.** *Find parametric equations for the tangent line to the curve given by the vector function* $\mathbf{r}(t) = \langle \ln(t+1), t\sin 2t, e^{-2t}\rangle$ *at the point* $(0,0,1)$.

$t=0$ corresponds to $(x,y,z) = (0,0,1)$.

A direction vector to the tangent line is

$$v = r'(t)\big|_{t=0} = \left\langle \frac{1}{t+1}, \sin 2t + 2t\cos 2t, -2e^{-2t} \right\rangle\Big|_{t=0}$$

$$= \langle 1, 0, -2 \rangle$$

An equation of the line: $\langle x,y,z\rangle = \langle 0,0,1\rangle + t\langle 1,0,-2\rangle$

$x = t, \; y = 0, \; z = 1-2t$

Definition: *The* ***curvature*** *of a curve given by the vector valued function* $\mathbf{r}$ *is*

$$\kappa(t) = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3},$$

where $\mathbf{T}$ *is the unit tangent vector.*

Example 16 (13.3)**.** *Consider a curve given by the vector function*

$$\mathbf{r}(t) = \left\langle t, , \frac{1}{t}, \sqrt{2}\ln t \right\rangle.$$

(a) Find the length of the curve from $(1,1,0)$ *to* $\left(e, \frac{1}{e}, \sqrt{2}\right)$.

$t=1$ $t=e$

$$r'(t) = \left\langle 1, -\frac{1}{t^2}, \frac{\sqrt{2}}{t} \right\rangle$$

$$\Rightarrow |r'(t)| = \sqrt{1 + \frac{1}{t^4} + \frac{2}{t^2}} = \frac{\sqrt{t^4+2t^2+1}}{t^2} = \frac{(t^2+1)}{t^2}$$

as $t^2+1 > 0$.

$$\text{Length} = \int_1^e |r'(t)|\,dt = \int_1^e (1+t^{-2})\,dt = \left[t - \frac{1}{t}\right]_1^e$$

$$= \left[\left(e - \frac{1}{e}\right) - (1-1)\right] = \frac{e^2-1}{e}$$

(b) *Find the unit tangent vector* $\mathbf{T}(t)$ *and the unit normal vector* $\mathbf{N}(t)$ *at the point* $(1,1,0)$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{t^2}{1+t^2}\left\langle 1, -\frac{1}{t^2}, \frac{\sqrt{2}}{t}\right\rangle$$

$$= \frac{1}{1+t^2}\left\langle t^2, -1, \sqrt{2}t\right\rangle$$

$$T(1) = \frac{1}{2}\left\langle 1, -1, \sqrt{2}\right\rangle$$

$$T'(t) = \frac{-2t}{(1+t^2)^2}\left\langle t^2, -1, \sqrt{2}t\right\rangle + \frac{1}{1+t^2}\left\langle 2t, 0, \sqrt{2}\right\rangle$$

$$T'(1) = -\frac{1}{2}\left\langle 1, -1, \sqrt{2}\right\rangle + \frac{1}{2}\left\langle 2, 0, \sqrt{2}\right\rangle = \frac{1}{2}\left\langle 1, 1, 0\right\rangle$$

$$N(1) = \frac{T'(1)}{|T'(1)|} = \frac{\frac{1}{2}\langle 1, 1, 0\rangle}{\frac{1}{2}\sqrt{1+0+1}} = \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle$$

(c) *Find the curvature of the curve at the point* $(1,1,0)$.

From above, $|T'(1)| = \frac{\sqrt{2}}{2}$ and

$|r'(1)| = 2$. So,

$$K(1) = \frac{|T'(1)|}{|r'(1)|} = \frac{\sqrt{2}/2}{2} = \frac{\sqrt{2}}{4}$$

Example 17 (13.4)**.** *The position function of a moving particle in space is given by* $\mathbf{r}(t) = \langle \sin t, 2t+1, \cos t \rangle$ *. Find its velocity, speed, and acceleration at time* $t = \pi$*.*

$v(t) = r'(t) = \langle \cos t, 2, -\sin t \rangle$

$v(\pi) = \langle -1, 2, 0 \rangle$

speed $= |v(\pi)| = \sqrt{5}$

$a(t) = v'(t) = \langle -\sin t, 0, -\cos t \rangle$

$a(\pi) = \langle 0, 0, 1 \rangle$

Example 18 (13.4)**.** *Find the velocity and position vector of a particle such that*

$$\mathbf{a}(t) = (-\cos t)\mathbf{i} + 2\mathbf{j} + 4e^{-2t}\mathbf{k}, \ \mathbf{v}(0) = -2\mathbf{k}, \ \mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

$v(t) = \int a(t)\,dt = \langle -\sin t, 2t, -2e^{-2t} \rangle + \vec{c}$

$\langle 0, 0, -2 \rangle \overset{\text{Given}}{=} v(0) = \langle 0, 0, -2 \rangle + \vec{c} \Rightarrow \vec{c} = 0.$

$\boxed{v(t) = \langle -\sin t, 2t, -2e^{-2t} \rangle}$

$r(t) = \int v(t)\,dt = \langle \cos t, t^2, e^{-2t} \rangle + \vec{d}$

$\langle 1, 3, 1 \rangle \overset{\text{Given}}{=} r(0) = \langle 1, 0, 1 \rangle + \vec{d}$

$\vec{d} = \langle 1, 3, 1 \rangle - \langle 1, 0, 1 \rangle = \langle 0, 3, 0 \rangle$

So, $\boxed{r(t) = \langle \cos t, t^2 + 3, e^{-2t} \rangle}$