Math 152 - Week-in-Review 8

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Exam 2: ① TOD, ② Int. Test, ③ CT, LCT, ④ AST, ⑤ RT, Power Series

Do the following series converge or diverge?
Which test should be used to demostrate convergence or divergence?

1. $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$. $= \sum a_n$

① TOD: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+n^2} = 0$

Test of Div fails.

② Int. test $a_n \to f(x) = \frac{1}{1+x^2}$

$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{1+x^2}dx = \arctan(x)\Big|_0^\infty$

$= \arctan(\infty) - \arctan(0)$

$= \frac{\pi}{2} - 0$

Since $\int_0^\infty f(x)dx$ is finite, $\sum_{n=0}^{\infty} a_n$ converges

Ans: Series converges by Integral Test.

③ Comparison Test also works

$b_n = \frac{1}{n^2}$ → converges and is larger.

2. $\sum_{n=1}^{\infty} \frac{1}{n+2^n}$. $= \sum a_n$

Comparison Test

$b_n = \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^n}}_{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}$

converging geometric series.

larger series

~~$c_n = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}$ diverges~~

larger series

a_n vs b_n

$n+2^n \geq 2^n$

$\frac{1}{n+2^n} \leq \frac{1}{2^n}$

a_n vs c_n

$n+2^n \geq n$

$\frac{1}{n+2^n} \leq \frac{1}{n}$

Ans: $\sum_{n=1}^{\infty} a_n$ converges by comparison test by comparing it to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ which converges. and is larger.

3. $\sum_{n=1}^{\infty} 7\sin\left(\frac{\pi}{n}\right)$. $= \sum a_n$

$b_n = ?? = \underbrace{\sum_{n=1}^{\infty} \frac{\pi}{n}(7)}$

Series diverges by p-series

$\boxed{\sin\left(\frac{\pi}{n}\right) \sim \frac{\pi}{n}}$

Th: $\lim_{x\to 0} \sin(x) \sim x$

$\lim_{n\to\infty} \frac{\pi}{n} = 0$

Limit Comparison Test

$\lim_{n\to\infty} \frac{a_n}{b_n} = L$

If $L>0$ but still finite, both $\sum a_n$ & $\sum b_n$ will do the same thing

$\lim_{n\to\infty} \frac{\cancel{7}\sin(\pi/n)}{\cancel{7}\,\pi/n} \sim \frac{0}{0}$

$\xrightarrow{L'H} \dfrac{\cos\left(\frac{\pi}{n}\right)\cdot\cancel{\left(-\frac{\pi}{n^2}\right)}}{\cancel{\left(-\pi/n^2\right)}} = \lim\limits_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$

$\therefore$ Both $\sum a_n$ and $\sum b_n$ will diverge.

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4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{(1/n)}}$. $= \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\frac{1}{n}}}$

TOD: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{2^{\frac{1}{n}}} = \dfrac{1}{2^{\frac{1}{\infty}}} = \dfrac{1}{2^0} = 1$

$\neq 0$

$\therefore$ Series diverges by Test of Divergence.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{1+7^n}$. $= \sum a_n \xrightarrow[\text{Test}]{\text{Comparison}} \sum b_n = \sum \dfrac{5^n}{7^n} = \sum \left(\frac{5}{7}\right)^n$

converging geometric series.
that is larger

$1+7^n \geq 7^n$

$\dfrac{1}{1+7^n} \leq \dfrac{1}{7^n}$

$\dfrac{5^n}{1+7^n} \leq \dfrac{5^n}{7^n}$

$\therefore$ By comparison test $\sum a_n$ also converges.

Q) what about ratio test?

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5n+1}$. $\longrightarrow$ Alternating Series Test.

a) $\lim\limits_{n\to\infty} |a_n| = 0$??

$|a_n| = \dfrac{n^2}{n^2+5n+1}$

$\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \dfrac{n^2}{n^2+5n+1} = \dfrac{\cancel{n^2}}{\cancel{n^2}} = 1$
$\neq 0$

TOD: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(-1)^n \cdot n^2}{n^2+5n+1} = (-1)^n \cdot \dfrac{\cancel{n^2}}{\cancel{n^2}} = (-1)^n$

Series diverges by Test of Divergence.

Series diverges by Test of Divergence.

7. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$. $= \sum_{n=1}^{\infty} \frac{(2n)!}{n!\, n!}$

factorials $\longrightarrow$ Ratio Test.

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

$L<1$ for convergence (absolute)

$a_n = \frac{(2n)!}{n!\,n!}$; $a_{n+1} = \frac{[2(n+1)]!}{(n+1)!(n+1)!} = \frac{(2n+2)!}{(n+1)!(n+1)!}$

RT: $\lim_{n\to\infty} \left| \frac{\cancel{(2n+2)!}^{(2n+1)(2n+2)}}{\cancel{(n+1)!}_{(n+1)}\cancel{(n+1)!}_{(n+1)}} \cdot \frac{\cancel{n!\,n!}}{\cancel{(2n)!}} \right| = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = \frac{4n^2}{n^2} = 4$

Since ratio $=4>1$, series diverges by Ratio Test

8. $\sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!}$.

$a_n = \frac{(-1)^n \cdot 3^n}{(2n+1)!}$ $\quad a_{n+1} = \frac{(-1)^{n+1} \cdot 3^{n+1}}{(2n+3)!}$

RT: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot \cancel{3^{n+1}}^{3}}{\cancel{(2n+3)!}_{(2n+2)(2n+3)}} \cdot \frac{\cancel{(2n+1)!}}{(-1)^n \cdot \cancel{3^n}} \right| = \lim_{n\to\infty} \frac{3}{(2n+2)(2n+3)} = 0$.

$\therefore$ Series converges absolutely by Ratio Test.

9. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot \underbrace{(3n+2)}}$.

$3(n+1)+2$

factorial $\rightarrow$ Ratio Test.

$a_1 = \frac{(-1) \cdot 2 \cdot (1!)}{5}$

$a_2 = \frac{(+1) \cdot 2^2 \cdot 2!}{5 \cdot 8}$

$a_3 = \frac{(-1) \cdot 2^3 \cdot 3!}{5 \cdot 8 \cdot 11}$

$\vdots$

$a_{n+1} = \frac{(-1)^{n+1}\, 2^{n+1}\, (n+1)!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)(3n+5)}$

RT: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot \cancel{2^{n+1}}^{2} \cdot \cancel{(n+1)!}^{(n+1)}}{\cancel{5 \cdot 8 \cdot 11} \cdots \cancel{(3n+2)}(3n+5)} \cdot \frac{\cancel{5 \cdot 8 \cdot 11} \cdots \cancel{(3n+2)}}{(-1)^n \cdot \cancel{2^n} \cdot \cancel{n!}} \right|$

$= \lim_{n\to\infty} \frac{2(n+1)}{3n+5} = \frac{2}{3}$

Since ratio is less than 1
Series absolutely converges by Ratio Test.

absolute convergence   conditional convergence

Do the following series converge absolutely, converge but not absolutely, or diverge?

10. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+1}{3^n}$. $= \sum a_n$

$b_n = |a_n| = \dfrac{n^2+1}{3^n}$

compare to $\dfrac{n^2}{3^n}$?

exp. term → Ratio test

RT: $\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)^2+1}{\cancel{3^{n+1}}_{3}}\cdot\frac{\cancel{3^n}}{n^2+1}\right|$

$\displaystyle = \lim_{n\to\infty}\frac{1}{3}\left[\frac{(n+1)^2+1}{n^2+1}\right] = \frac{1}{3}\cdot\frac{\cancel{n^2}}{\cancel{n^2}}$

Ratio < 1

$\therefore \sum b_n$ converges.

Ans: Since $\sum b_n$ converges,

$\sum a_n$ converges absolutely.

11. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)}$. $= \sum a_n$

alternating series test

① $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n\ln(n)} \Rightarrow \frac{1}{\infty} = 0$.

② Since $n\ln(n)$ is increasing

$a_n = \dfrac{1}{n\ln(n)}$ is decreasing

$\therefore \sum a_n$ converges by AST.

$\therefore \sum a_n$ is conditionally convergent.

$\displaystyle\sum b_n = |a_n| = \sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$

① TOD fails

② Integral test

$a_n \to f(x) = \dfrac{1}{x\ln(x)}$

$\displaystyle\int_2^{\infty} f(x)\,dx = \int \frac{1}{x\ln(x)}\,dx = \int\frac{du}{u}$

$= \ln|u|$

$= \ln|\ln(x)|\Big|_2^{\infty}$

$= \ln|\ln(\infty)| - \ln|\ln 2|$

$\to \infty$, hence diverges

$-1 \le \cos(n) \le 1$

$0 \le |\cos(n)| \le 1$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{3n+2}$. $= \sum a_n$

$\sum b_n = \sum |a_n| = \sum \left| \frac{\cos(n\pi)}{3n+2} \right| = \frac{[0,1]}{3n+2}$

$\sum b_n$ diverges by comparison to $\sum \frac{1}{3n}$ by LCT.

$\sum a_n = \frac{[-1, 1]}{3n+2}$

$\sum a_n$ also diverges by comparison to $\sum \frac{1}{3n}$ by LCT.

$\therefore \sum a_n$ diverges.

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n^2+3}$. $= \sum a_n$

$\sum b_n = \sum |a_n| = \sum_{n=1}^{\infty} \frac{n}{2n^2+3}$

diverges by comparison test comparing to $\sum_{n=1}^{\infty} \frac{1}{2n}$ (LCT)

~~or~~ or by Integral Test.

$\int_1^{\infty} \frac{x}{2x^2+3}\,dx = \int \frac{du/4}{u}$

$= \frac{1}{4} \ln(2x^2+3) \Big|_1^{\infty}$

$\to \infty$

$\sum b_n$ diverges

AST.

a) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{2n^2+3} = 0$

b) $\frac{1}{5} > \frac{2}{11}$ (decreasing)

converge by AST.

$\therefore \sum a_n$ is conditionally convergent.

$a_n = \frac{n}{2n^2+3} \to f(x) = \frac{x}{2x^2+3}$

$f'(x) = \frac{(2x^2+3)(1) - (x)(4x)}{(2x^2+3)^2} = \frac{2x^2+3-4x^2}{(2x^2+3)^2} = \frac{3-2x^2}{(2x^2+3)^2}$

$f'(x) = \frac{3-2x^2}{(2x^2+3)^2}$ → square is always (+)ve

$n=1$ $3-2n^2 = 3-2 = 1 \to +$

$n=2$ $3-2(2^2) = -5 \to -$

$f'(x)$ is (−)ve for $n \ge 2$

Alternating Series Remainder Estimate.

14. How many terms are required to approximate the sum of the series $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}}_{\text{Alternating series}}$ so that the error is less than 0.001?

$|R_n| \leq a_{n+1}$

$a_n = \frac{(-1)^{n+1}}{n^4}$

$R_n \leq \frac{1}{(n+1)^4}$

$a_{n+1} = \frac{(-1)^{n+2}}{(n+1)^4}$

$\frac{1}{(n+1)^4} \leq 0.001$

$(n+1)^4 \geq \frac{1}{0.001} = 1000$

$(n+1) \geq \sqrt[4]{1000} \approx 5.623$

$-1 \qquad -1$

$n \geq 4.623$

$n=5$ $\therefore$ Ans: You need 5 terms.

Alternate question error <0.0001

then, $(n+1)^4 \geq \frac{1}{0.0001} = 10{,}000$

$(n+1) \geq \sqrt[4]{10{,}000} = 10$

$n \geq 9$

You'd need 9 terms.

15. Approximate the sum of the series $\sum_{n=1}^{\infty}(-1)^n ne^{-2n}$ to within 4 decimal places.

0.0001

$|R_n| \leq a_{n+1}$

$(n+1)e^{-2(n+1)} \leq 0.0001$

Trial & Error method.

$\frac{n+1}{e^{2n+2}} \leq 0.0001$

$S = S_5 + R_5 = S_5 \pm |a_6|$

$n=1 \quad \frac{2}{e^4} = 0.03663$

$n=2 \quad \frac{3}{e^6} = 0.00743$

$n=3 \quad \frac{4}{e^8} = 0.0013$

$n=4 \quad \frac{5}{e^{10}} = 0.000227$

$n=5 \quad \frac{6}{e^{12}} = 0.0000363$

Ans: You need 5 terms

$\sum_{n=0}^{\infty} c_n (x-a)^n$ → power series with center @ $x=a$.

$a-R \leftarrow \underbrace{\overset{R}{\longleftarrow}\; x=a \;\overset{R}{\longrightarrow}}_{IC} \rightarrow a+R$

11.8

Find the Radius (R) and the Interval of Convergence (IC) for the following power series. → Ratio Test

16. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{2^n}$.

$(x+2)=0$ center $x=-2=a$

RT: $\displaystyle\lim_{n\to\infty}\left|\frac{\cancel{(x+2)^{n+1}}}{\cancel{2^{n+1}}_{2}}\cdot\frac{\cancel{2^n}}{\cancel{(x+2)^n}}\right| = \left|\frac{x+2}{2}\right| < 1$ for series to converge.

Solve for x.

$-1 < \dfrac{x+2}{2} < 1$

$-2 < x+2 < 2$

$-4 < x < 0$

$\xleftarrow{2}\; \xrightarrow{2}$: -4, $x=-2$, 0

$R = \dfrac{0-(-4)}{2} = \dfrac{4}{2} = 2$

IC: $(-4, 0)$

Test $x=-4$

$\displaystyle\sum_{n=0}^{\infty}\frac{(-4+2)^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot\cancel{2^n}}{\cancel{2^n}} = \sum_{n=0}^{\infty}(-1)^n$

diverges

Test $x=0$

$\displaystyle\sum_{n=0}^{\infty}\frac{(0+2)^n}{2^n} = \sum_{n=0}^{\infty}\frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$

diverges

17. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!(x-5)^n}{2n+1}$.

center: $x-5=0$ or $a=5$

RT: $\displaystyle\lim_{n\to\infty}\left|\frac{\overset{(2n+1)(2n+2)}{\cancel{(2n+2)!}}\ \overset{(x-5)}{\cancel{(x-5)^{n+1}}}}{(2n+3)}\cdot\frac{(2n+1)}{\cancel{(2n)!}\ \cancel{(x-5)^n}}\right|$

$\displaystyle = \underbrace{|x-5|}\ \lim_{n\to\infty}\frac{(2n+1)(2n+1)(2n+2)}{2n+3}$ → $\displaystyle\lim_{n\to\infty}\frac{8n^3}{2n} = 4n^2 \to \infty$

force $|x-5|=0$. ∴ $x=5$ exactly → Point converges

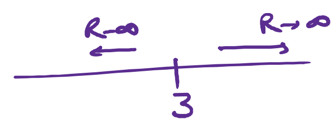
Series only converges at $x=5$

$R=0$ IC: $\{5\}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{3^n(x-1)^n}{n\ln(n)}$.

19. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{n!}$.

center at $x=3$



RT: $\displaystyle\lim_{n\to\infty} \left| \frac{\cancel{2^{n+1}}^{\,2}\,\cancel{(x-3)^{n+1}}^{\,(x-3)}}{\cancel{(n+1)!}_{\,(n+1)}} \cdot \frac{\cancel{n!}}{\cancel{2^n}\,\cancel{(x-3)^n}} \right|$

$\displaystyle = |2(x-3)| \lim_{n\to\infty} \left(\cancel{\frac{1}{n+1}}\right) \to 0$

ratio $= 0$ for all values of x

$R \to \infty$

IC: $(-\infty, \infty)$

20. If the power series given by $\sum_{n=0}^{\infty} C_n(x-2)^n$ coverges at $x=5$ and diverges at $x=-4$, what can we say about the following?

(a) $\sum_{n=0}^{\infty} C_n$

(b) $\sum_{n=0}^{\infty} C_n(-3)^n$

(c) $\sum_{n=0}^{\infty} C_n 9^n$

(d) $\sum_{n=0}^{\infty} C_n(-5)^n$

(e) $\sum_{n=0}^{\infty} C_n(-2)^n$

(f) $\sum_{n=0}^{\infty} C_n 4^n$