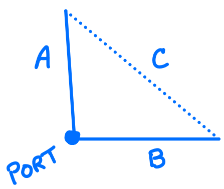
Session 6: Sections 2-6 (Related Rates), 3-1, and 3-2

1. Two ships leave the same port at the same time. One hour after departure, ship A is 3 miles north of port traveling north at 6 miles per hour and ship B is 4 miles east of port traveling back to port at 7 miles per hour. At what rate is the distance between the two ships changing one hour after departure?

Step 1: Draw a Picture - Sketch a picture that represents the problem and label any variables of interest.



A is distance ship A is from Port
B is distance ship B is from Port
C is distance ship A + B are from each other

Step 2: State Given & To Be Determined Information - Identify (a) the given (or known) values for any variable or rate of change, (b) the rates of change that need to be determined and at the specific values at which the rates need to be determined. Be sure to include all units.

Known: $\frac{dA}{dt} = 6 \text{ m/hr}$ when $A = 3$

$\frac{dB}{dt} = -7 \text{ m/hr}$ when $B = 4$

TBD: $\frac{dC}{dt}$

Step 3: Formulate Equations - Look at the information collected in steps 1 and 2 to find a formula that represents the relationships between the rates and the variables.

$$A^2 + B^2 = C^2$$

Step 4: Take Derivative - Use implicit differentiation to take a derivative of the equation in step 3 with respect to time, t.

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\frac{dC}{dt}$$

divide equation by 2

$$A\frac{dA}{dt} + B\frac{dB}{dt} = C\frac{dC}{dt}$$

Step 5: Substitute - Substitute all known values (see step 2) into the equation found in step 4, and then solve for the unknown rate of change.

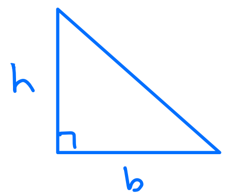
$$3(6) + 4(-7) = 5\frac{dC}{dt}$$
$$18 - 28 = 5\frac{dC}{dt}$$
$$-10 = 5\frac{dC}{dt}$$
$$-2 = \frac{-10}{5} = \frac{dC}{dt}$$

Find C:
$$3^2 + 4^2 = C^2$$
$$9 + 16 = C^2$$
$$25 = C^2$$
$$5 = C$$

$\boxed{\text{The distance between the ships is decreasing at 2 m/hr one hour after departure.}}$

2. The base of a right triangle is growing at a rate of 3 inches per hour, and its height is shrinking at a rate of 3 inches per hour. When the base of the triangle has a length of 12 inches and the height is 16 inches, how fast is the area of the triangle changing?

STEP 1:



b is the length of the base of the triangle
h is the height of the triangle.
A is the area of the triangle

STEP 2:

Known:

$\frac{db}{dt} = 3 \frac{in}{hr}$

$\frac{dh}{dt} = -3 \frac{in}{hr}$

TBD:

$\frac{dA}{dt}$ when $b = 12in$ and $h = 16in.$

STEP 3+4:

$A = \frac{1}{2} bh$

$\frac{dA}{dt} = \frac{1}{2} b \frac{dh}{dt} + \frac{1}{2} \frac{db}{dt} h$

STEP 5:

$\frac{dA}{dt} = \frac{1}{2}(12)(-3) + \frac{1}{2}(3)(16) \longrightarrow \frac{dA}{dt} = -18 + 12 = -6$

$\boxed{\text{The area is decreasing by } 6 \frac{in^2}{hr.}}$

3. A manufacturer determines the relationship between the demand and price for an item is given by $3x^2 + 3xp + 40p^2 = 160,000$, where x is the number of items demanded each month at a price of p dollars each. If the number of items demanded is decreasing at a rate of 10 items per month, find the rate of change of the price with respect to time when demand is 200 items each month.

STEP 1: N/A

STEP 2:

Known: $\frac{dx}{dt} = -10 \frac{items}{month}$

TBD: $\frac{dp}{dt}$ when $x = 200$

STEP 3+4: $3x^2 + 3xp + 40p^2 = 160000$

$6x \frac{dx}{dt} + \underbrace{3x \frac{dp}{dt} + 3 \frac{dx}{dt} p}_{\text{product rule for } 3xp} + 80p \frac{dp}{dt} = 0$

Step 5:

Find p when $x = 200$

$3(200)^2 + 3(200)p + 40p^2 = 160000$

$3(40000) + 600p + 40p^2 = 160000$

$120000 + 600p + 40p^2 = 160000$

$40p^2 + 600p - 40000 = 0$

$40(p^2 + 15p - 1000) = 0$

$40(p+40)(p-25) = 0$

$\cancel{p = -40}$ $\boxed{p = 25}$

$6(200)(-10) + 3(200)\frac{dp}{dt} + 3(-10)(25) + 80(25)\frac{dp}{dt} = 0$

$-12000 + 600 \frac{dp}{dt} - 750 + 2000 \frac{dp}{dt} = 0$

$2600 \frac{dp}{dt} = 12750$

$\boxed{\frac{dp}{dt} = 4.9 \frac{dollars}{month}}$

Finding Increasing and Decreasing Intervals

Step 1: Find all partition numbers of $f'(x)$ and then determine which partition numbers are critical values of $f(x)$.

Step 2: Plot all partition numbers on the sign chart template below. Put a solid dot on all critical values and an open circle on all partition numbers that are not critical values.

Step 3: Select one x-value in each interval, and evaluate $f\,'(x)$ at each selected x-value to determine whether $f\,'(x)$ is positive or negative on each interval. Indicate whether $f\,'(x)$ is positive or negative on each interval by writing "+" or "−" on the top of the sign chart.

Step 4: Apply the Increasing/Decreasing Test. Recall that where $f\,'(x)$ is "+", then $f(x)$ is increasing ($\nearrow$). Where $f\,'(x)$ is "−", then $f(x)$ is decreasing ($\searrow$).

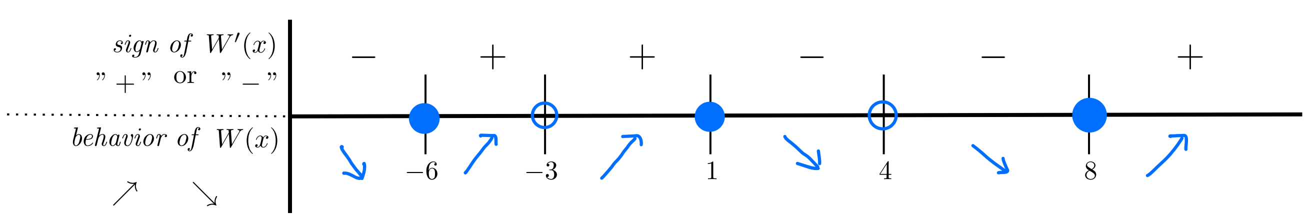
4. President Welsh's favorite function is $W(x)$ with a domain of $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$. He is interested in determining some characteristics of his favorite function and begins by taking the first derivative. In doing so he finds the partition numbers. More specifically he found $W'(x)=0$ when $x=-6,1$ and $f'(x)$ DNE when $x=-3,4,8$.

   (a) Which of these partition numbers are critical values? How do you know this?

   partition $\longrightarrow$ $x = -6, -3, 1, 4\ 8$

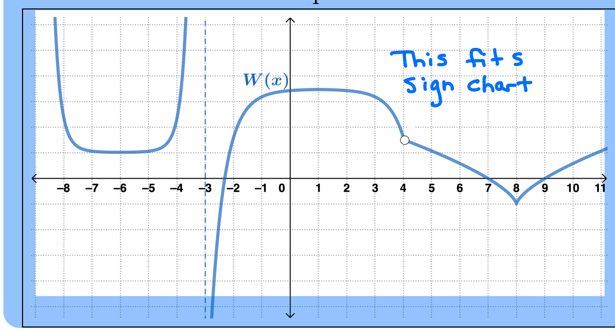
   Critical $\longrightarrow$ $x = -6, 1, 8$ because these are the partition numbers in the domain of $w(x)$.

   (b) President Welsh started a first derivative sign chart, but needs your help to complete it. Fill in all missing informaiton to make the first derivative sign chart complete.

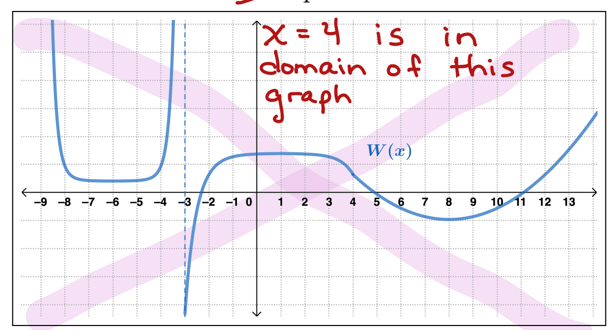


(c) Which images below could be a graph of $W(x)$?

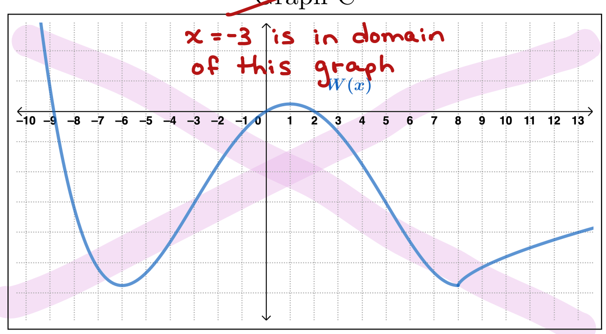
Graph A



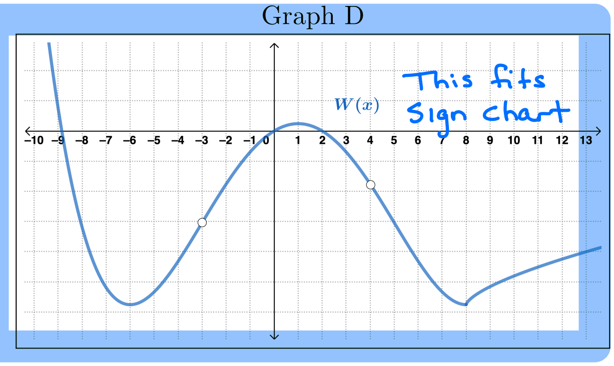
~~Graph~~ B



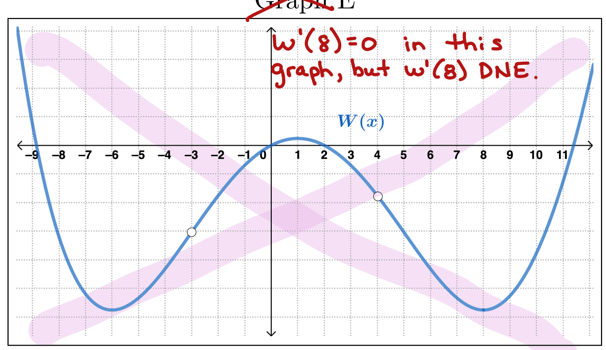
~~Graph~~ C



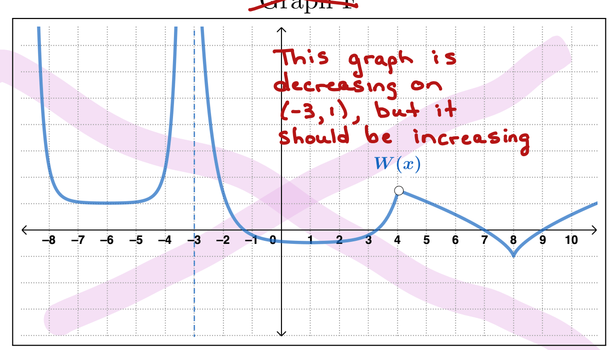
Graph D



~~Graph~~ E



~~Graph~~ F



Local Extrema
Suppose $f(x)$ is continuous on an interval containing the critical value $x = c$ and $f(x)$ is differentiable near, and on both sides of, $x = c$. Then, $f(c)$ satisfies one of the following:

Local Max: If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(c)$ is a local maximum of $f(x)$.

Local Min: If $f'(x)$ changes sign from negative to positive at $x = c$, then $f(c)$ is a local minimum of $f(x)$.

Neither: If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is neither a local maximum nor a local minimum of $f(x)$.

5. Given $f'(x) = (x-4)^2(x-1)(x+2)^3(x+5)$ and that the domain of $f(x)$ is $(-\infty, \infty)$, find all (a) partition numbers, (b) critical values, (c) the intervals where $f(x)$ is increasing, (d) the intervals where $f(x)$ is decreasing, and (e) all local extrema.

(a) The partition numbers of $f'(x)$ are $x = 4, 1, -2, -5$

(b) The critical values of $f(x)$ are $x = 4, 1, -2, -5$

(c) $f(x)$ is increasing on $(-5,-2), (1,4), (4,\infty)$

(d) $f(x)$ is decreasing on $(-\infty,-5), (-2,1)$

(e) $f(x)$ has local max at $x =$ -2

(f) $f(x)$ has local min at $x =$ $-5, 1$

Sign Chart Template - First Derivative

sign of $f'(x)$ " + " or " − "

$-$ $+$ $-$ $+$ $+$

behavior of $f(x)$ ↗ ↘

↘ -5 min   ↗ -2 max   ↘ 1 min   ↗ 4 ↗

Test values: -6, -3, 0, 2, 5

$\underbrace{(-6-4)^2(-6-1)(-6+2)^3(-6+5)}_{+\ -\ -\ -}$ neg

$\underbrace{(-3-4)^2(-3-1)(-3+2)^3(-3+5)}_{+\ -\ -\ +}$ pos

$\underbrace{(0-4)^2(0-1)(0+2)^3(0+5)}_{+\ -\ +\ +}$ neg

$\underbrace{(2-4)^2(2-1)(2+2)^3(2+5)}_{+\ +\ +\ +}$ pos

$\underbrace{(5-4)^2(5-1)(5+2)^3(5+5)}$ pos

6. Given $f(x) = \dfrac{(x-1)^2}{x+2}$, $f'(x) = \dfrac{(x-1)(x+5)}{(x+2)^2}$, and $\overbrace{f''(x) = \dfrac{18}{(x+2)^3}}^{\text{extra}}$, find all (a) partition numbers, (b) critical values, (c) the intervals where $f(x)$ is increasing, (d) the intervals where $f(x)$ is decreasing, and (e) all local extrema. Be sure to draw a sign chart for the first derivative.

(a) The partition numbers of $f'(x)$ are $x = -5, -2, 1$

(b) The critical values of $f(x)$ are $x = 1, -5$

(c) $f(x)$ is increasing on $(-\infty, -6), (1, \infty)$

(d) $f(x)$ is decreasing on $(-5, -2), (-2, 1)$

(e) $f(x)$ has local max at $x = -5$

(f) $f(x)$ has local min at $x = 1$

$f'(x) = 0$

$0 = (x-1)(x+5)$

$x = 1, -5$

$f'(x)$ DNE when

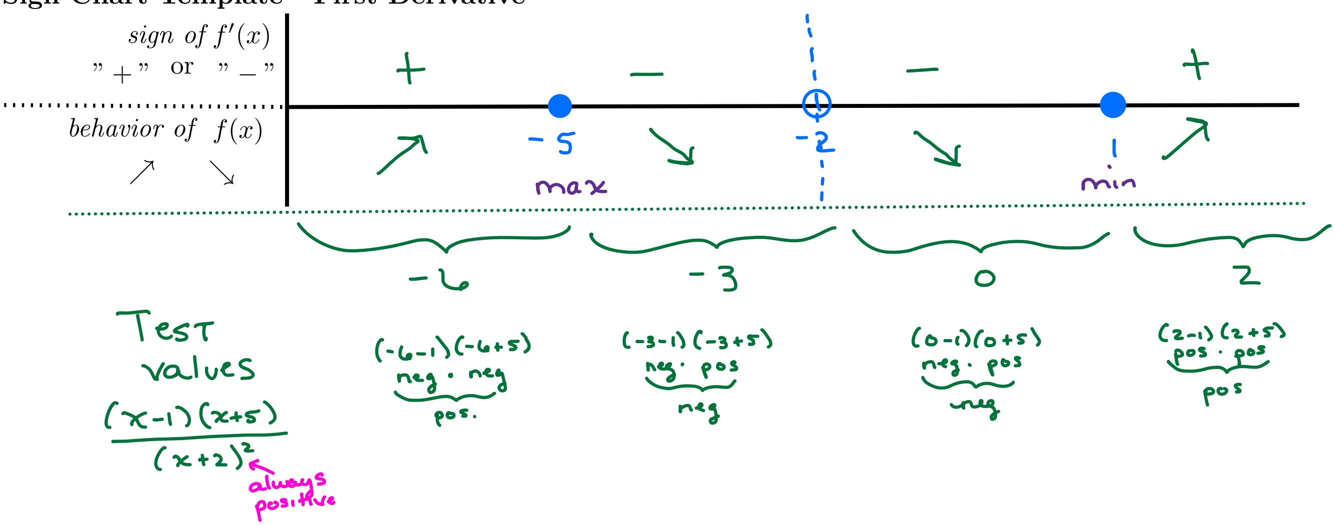
$x + 2 = 0$

$x = -2$

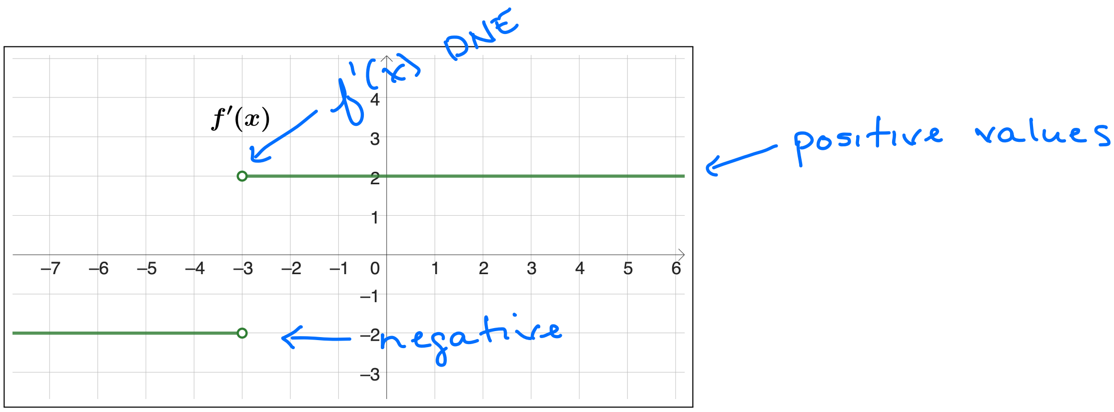
Domain

$(-\infty, -2) \cup (-2, \infty)$

Sign Chart Template - First Derivative



7. Given the graph of $f'(x)$ below and that the domain of $f(x)$ is $(-\infty, \infty)$, find all (a) partition numbers, (b) critical values, (c) the intervals where $f(x)$ is increasing, (d) the intervals where $f(x)$ is decreasing, and (e) all local extrema.



(a) The partition numbers of $f'(x)$ are $x = -3$

(b) The critical values of $f(x)$ are $x = -3$

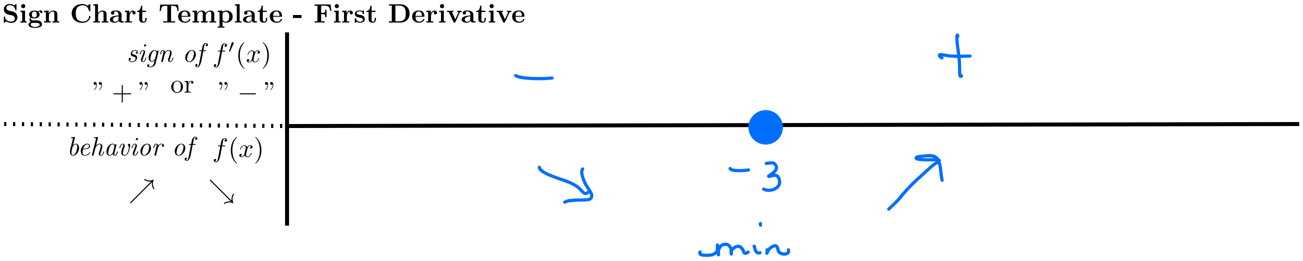
(c) $f(x)$ is increasing on $(-3, \infty)$

(d) $f(x)$ is decreasing on $(-\infty, -3)$

(e) $f(x)$ has local max at $x =$ none

(f) $f(x)$ has local min at $x =$ -3

Sign Chart Template - First Derivative



Finding Intervals of Concavity and Inflection Points

Step 1: Find all partition numbers of $f'(x)$ and then determine which partition numbers are critical values of $f(x)$.

Step 2: Plot all partition numbers on the $f''(x)$ sign chart (see template below). Indicate whether each partition number is in the domain of $f(x)$ by drawing a solid dot and an open circle if the partition number is not in the domain.

Step 3: Select one x-value in each interval, and evaluate $f''(x)$ at each selected x-value to determine whether $f''(x)$ is positive or negative on each interval. Indicate whether $f''(x)$ is positive or negative on each interval by writing "+" or "−" on the top of the sign chart.

Step 4: Apply the Concavity Test. Recall that where $f''(x)$ is "+", then $f(x)$ is concave up ($\smile$). Where $f''(x)$ is "−", then $f(x)$ is concave down ($\frown$).

Definition of Inflection Point: If $f(x)$ is continuous and changes concavity at a point, then the point is called an inflection point.

8. Given $f(x) = \frac{1}{2}e^x\left(x^2 - 4x - 10\right)$, $f'(x) = \frac{1}{2}e^x(x^2 - 2x - 14)$, and $f''(x) = \frac{1}{2}e^x(x^2 - 16)$, find the intervals of concavity and any inflection points of $f(x)$.

Domain $(-\infty,\infty)$

$f''(x) = 0$ when $\cancel{\frac{1}{2}e^x = 0}$ never

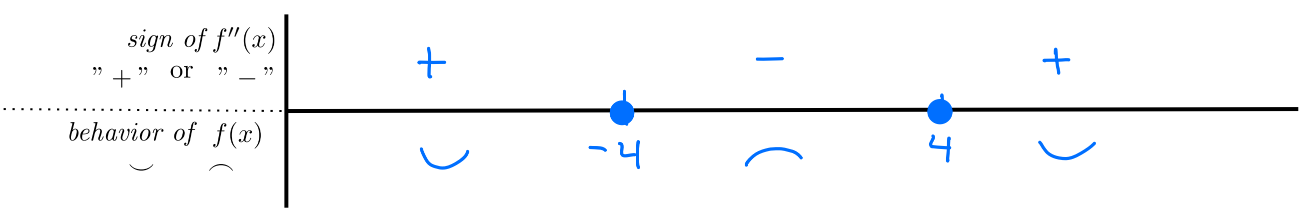
$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

$f''(x)$ exists everywhere

Sign Chart Template - Second Derivative



(a) $f(x)$ is concave up on $(-\infty,-4), (4,\infty)$

(b) $f(x)$ is concave down on $(-4, 4)$

(c) $f(x)$ has inflection points at $x =$ $-4, 4$

Done in Calculator

Test values	$\frac{1}{2}e^x(x^2-16)$
−5	pos
0	neg
5	pos

9. Given the information about $g(x)$ below, find the intervals where $g(x)$ is concave up and concave down. Then find the x-values where $g(x)$ has point(s) of inflection.

- The domain of $g(x)$ is $(-\infty, -2) \cup (-2, \infty)$.

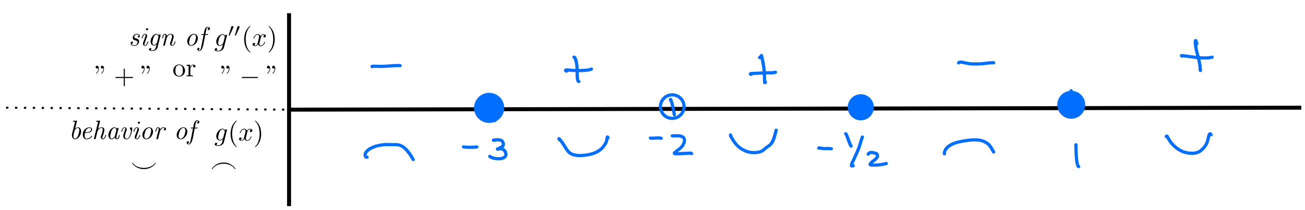
- $g'(x) = \dfrac{(x-7)(x+3)^2}{x+2}$

- $g''(x) = \dfrac{(x-1)(x+3)(2x+1)}{(x+2)^2}$ $\longrightarrow$ $g''(x)$ DNE $x=-2$

$g''(x) = 0$ when $x-1=0$ $x+3=0$ $2x+1=0$

$x=1$ $x=-3$ $x=-\frac{1}{2}$

Sign Chart Template - Second Derivative



(a) $g(x)$ is concave up on $(-3,-2), (-2,-\frac{1}{2}), (1,\infty)$

(b) $g(x)$ is concave down on $(-\infty,-3), (-\frac{1}{2}, 1)$

(c) $g(x)$ has inflection points at $x =$ $-3, -\frac{1}{2}, 1$

In calculator test values

$-4 \rightarrow$ neg

$-2.5 \rightarrow$ pos

$-1 \rightarrow$ pos

$.75 \rightarrow$ neg

$2 \rightarrow$ pos

The Second Derivative Test Suppose $f(x)$ is twice-differentiable at $x = c$ and $x = c$ is a critical value of $f(x)$ such that $f'(c) = 0$.

- If $f''(c) < 0$ then $f(x)$ is concave down and has a local maximum at $x = c$.
- If $f''(c) > 0$ then $f(x)$ is concave up and has a local minimum at $x = c$.
- If $f''(c) = 0$ then the Second Derivative Test fails, and $f(x)$ may have a local maximum, minimum, or neither at $x = c$.

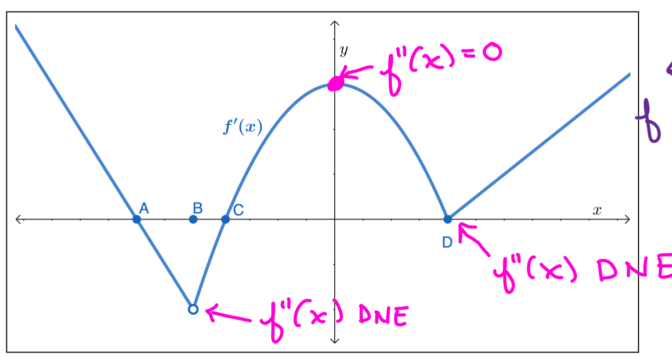
10. Given $f'(1) = 0$ and $f''(x) = -6x - 6$, use the second derivative test to determine if there is a local maximum or minimum at $x = 1$. (assume domain to be $(-\infty, \infty)$)

$f'(1) = 0$ means $x = 1$ is critical value

$f''(1) = -6(1) - 6 = -6 - 6 = -12 < 0$ $\quad x = 1$

$\boxed{x = 1 \text{ is local max}}$

11. Given the graph of f' below and that f is continuous on its domain of $(-\infty, \infty)$, find the following:



f: B 0 D

(a) the partition numbers of f''. $x = B, 0, D$

f	+	−	↗	↘	∪	∩
f'	+	−	↗	↘	∪	∩
f''	+	−	↗	↘	∪	∩

(b) the intervals of concavity of f.

f': ↘ | ↗ | ↘ | ↗

f: ∩ B ∪ 0 ∩ D ∪

Concave up $(B, 0), (D, \infty)$

Concave down $(-\infty, B), (0, D)$

(c) the x-values of any inflection points of f.

$x = B, 0, D$

(d) the intervals where f is increasing.

f': + − − + −

f: ↗ A ↘ B ↘ C ↗ D ↘

f is increasing $(-\infty, A), (C, D)$

12. Determine all local extrema and points of inflection for $f(x) = \frac{3}{5}x^5 - 8x^3$.

Domain $(-\infty, \infty)$

$$f'(x) = 3x^4 - 24x^2 = 3x^2(x^2-8)$$

$f'(x) = 0$

$3x^2 = 0 \quad x^2 - 8 = 0$

$x = 0 \quad x = \pm\sqrt{8}$

	max			min
sign f'	$+$	$-$	$-$	$+$
	$-\sqrt{8}$	0	$\sqrt{8}$	
behavior f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$
test	-3	-1	1	3
	pos	neg	neg	pos

$\boxed{\text{local max } x = -\sqrt{8} \qquad \text{local min } x = \sqrt{8}}$

$$f''(x) = 12x^3 - 48x = 12x(x^2-4)$$

$f''(x) = 0$

$12x = 0 \quad x^2 - 4 = 0$

$x = 0 \quad x = \pm 2$

sign f''	$-$	$+$	$-$	$+$
	-2	0	2	
behav f	$\frown$	$\smile$	$\frown$	$\smile$
test value	-3	-1	1	3
	neg	pos	neg	pos

$\boxed{\text{points of inflection } (-2, 44.8) \quad (0,0) \quad (2, -44.8)}$

$f(-2) = 44.8$

$f(0) = 0$

$f(2) = -44.8$