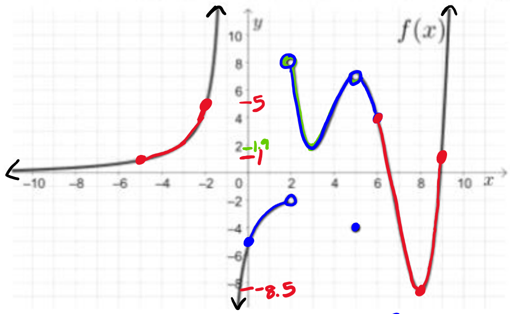
WIR#8

Week-in-Review#8 - Sections 3.4 and 3.5
Math 142 - Fall 2024, by K. Kilmer

Section 3.4

- **Definition of Absolute Extrema:**
  - $f(c)$ is an **absolute maximum** of f if $f(c) \geq f(x)$ for all x in the domain of f.
  - $f(c)$ is an **absolute minimum** of f if $f(c) \leq f(x)$ for all x in the domain of f.
  - $f(c)$ is an **absolute extremum** of f if $f(c)$ is an absolute maximum or absolute minimum.
- **Extreme Value Theorem:** If a function is continuous on a closed interval $[a, b]$, then it must have both an absolute maximum and absolute minimum on that interval.
- **The Closed Interval Interval Method:** This method is used to find the absolute extrema of a continuous function on a closed interval, $[a, b]$.
  - Find the critical values on the interval (a, b).
  - Evaluate f at the endpoints, a and b, and at the critical values that are on the interval.
  - The largest value obtained from the previous step is the absolute maximum of $f(x)$ on $[a, b]$ and the smallest value obtained is the absolute minimum of $f(x)$ on $[a, b]$.
- **The First Derivative Test for Absolute Extrema:** This method is used to find the absolute extrema of a continuous function, $f(x)$, on an interval (open, closed, half-open) that has only **ONE** critical value, $x = c$.
  - If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is an **absolute maximum** of $f(x)$.
  - If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is an **absolute minimum** of $f(x)$.
- **The Second Derivative Test for Absolute Extrema:** This method is used to find the absolute extrema of a continuous function, $f(x)$, on an interval (open, closed, half-open) that has only **ONE** critical value, $x = c$, for which $f'(c) = 0$.
  - If $f''(c) < 0$, then $f(c)$ is an **absolute maximum** of $f(x)$.
  - If $f''(c) > 0$, then $f(c)$ is an **absolute minimum** of $f(x)$.
- Note: For the **First Derivative Test for Absolute Extrema** and the **Second Derivative Test for Absolute Extrema**, if you are dealing with an **open** interval and you make a conclusion about the absolute extrema at the critical value, you can conclude that the other absolute extrema does not exist. For instance, if you are dealing with an open interval and conclude that there is an absolute minimum at $x = c$ (the only critical value on the interval), you can conclude that that function does not have an absolute maximum on the interval. The same idea holds in the opposite direction.

1. Find the absolute extrema of the function graphed below on each of the following intervals.



(a) $(-\infty, \infty)$ No abs max
No abs min

(b) $[-5, -2]$ Abs max of 5 at $x=-2$
Abs min of 1 at $x=-5$

(c) $(2, 5)$ No abs max
Abs min of 1.9 at $x=3$

(d) $[0, 6)$ No abs max
Abs min of -5 at $x=0$

(e) $[6, 9]$ Abs max of 4 at $x=6$
Abs min of -8.5 at $x=8$

2. Find the absolute extrema of $f(x) = \frac{3}{4}x^4 + \frac{19}{3}x^3 - 7x^2 + 30$ on each of the intervals given below:

Y_1

Domain: $(-\infty, \infty)$
Poly $\Rightarrow$ Always continuous

$f'(x) = 3x^3 + 19x^2 - 14x$
$= x(3x^2 + 19x - 14)$
$= x(3x-2)(x+7)$

Check: FOIL!
$(3x-2)(x+7)$
$= 3x^2 + 21x - 2x - 14$
$= 3x^2 + 19x - 14$ ✓

$f'(x)$ is always defined!

$f'(x) = 0$ when $x(3x-2)(x+7) = 0$

$x = 0$ $\quad$ $3x-2=0$ $\quad$ $x+7=0$
$3x = 2$ $\quad$ $x=-7$
$x = 2/3$

critical values of f!

(a) $[-5, 1]$ cont. function on a closed int $\Rightarrow$ Use Closed Interval method!
$x=0$ & $x=2/3$ are the CV on the int.

x	$f(x)$
-5	$\frac{-5615}{12} \approx -467.9$
0	30
$2/3$	$\frac{2342}{81} \approx 28.9$
1	$\frac{361}{12} \approx 30.1$

An abs max of $\frac{361}{12}$ occurs at $x=1$
An abs min of $\frac{-5615}{12}$ occurs at $x=-5$

(b) $[3, 4]$ cont. fcn on a closed int. ⇒ use C.I.M!

no CV on int ⇒

x	$f(x)$
3	198.75 ← Abs min
4	$\frac{1546}{3} \approx 515.3$ ← Abs max

Abs max of $\frac{1546}{3}$ at $x=4$
Abs min of 198.75 at $x=3$

(c) $(-\infty, -6)$ $x=-7$ is the only CV on this interval

Not a closed interval but f is cont.; we have one CV ⇒ Use FDT or SDT! $f(-7)$

Recall, $f'(x) = 3x^3+19x^2-14x$

$f''(x) = 9x^2+38x-14$

$f''(-7) = 9(-7)^2+38(-7)-14 = (+)$

⇒ Abs min of $\frac{-8215}{12}$ at $x=-7$
NO Abs max

(d) $(5, 8)$ NO CVs on our int.

NO Abs max
NO Abs min

3. Find the absolute extrema of $g(x) = \frac{1}{3}x - \sqrt{x+2}$ on each of the intervals given below:

$= \frac{1}{3}x - (x+2)^{1/2}$

Domain: $x+2 \geq 0$, $x \geq -2$ $[-2, \infty)$

cont on $[-2,\infty)$

$g'(x) = \frac{1}{3} - \frac{1}{2}(x+2)^{-1/2} \cdot 1$

$= \frac{1}{3} - \frac{1}{2(x+2)^{1/2}}$

$\frac{2(x+2)^{1/2}}{2(x+2)^{1/2}} \cdot \frac{1}{3} - \frac{1}{2(x+2)^{1/2}} \cdot \frac{3}{3} = \frac{2(x+2)^{1/2}-3}{6(x+2)^{1/2}}$

$g'(x)=0$ when $2(x+2)^{1/2}-3=0$

$2(x+2)^{1/2}=3$

$((x+2)^{1/2})^2 = \left(\frac{3}{2}\right)^2$

$x+2 = \frac{9}{4}$

$-2 \quad -2$

$x = \frac{1}{4}$

$g'(x)$ is undefined when $6(x+2)^{1/2}=0$

$((x+2)^{1/2})^2 = (0)^2$

$x+2=0$

$x=-2$

(a) $[-1, 1]$ cont. fcn on a closed int ⇒ use CIM!

$x=1/4$ is the only CV

x	$g(x)$
-1	$-4/3 \approx -1.333$ ← Abs max
$1/4$	$-17/12 \approx -1.417$ ← Abs min
1	$\frac{1}{3} - \sqrt{3} \approx -1.399$

Abs max of $-4/3$ at $x=-1$
Abs min of $-17/12$ at $x=1/4$

$g'(x) = \dfrac{2(x+2)^{1/2} - 3}{6(x+2)^{1/2}}$ y_1

(b) $(0, 2)$ g is cont. on this int. We have one CV ($x = 1/4$) and open int.
$\Rightarrow$ use FDT or SDT. I will use FDT.

g(x): ↘ ↗

g'(x): − | + on 0 ($x=0.2$), 1/4, ($x=1$), 2

g(1/4)

$\Rightarrow$ Abs min of $\frac{-17}{12}$ at $x = 1/4$
NO abs max

Section 3.5

Strategy for Solving Optimization Problems

1. Translate the word problem into mathematical form.
2. State the objective function in terms of one variable.
3. Find the interval on which the objective function must be optimized.
4. Use Calculus (the Closed Interval Method, First Derivative Test, or Second Derivative Test from Section 3.4) to find the optimal solution.
5. Answer the original question.

maximize or minimize | A function of one variable | on an interval

4. Find two positive numbers x and y whose product is 250 and for which $4x + 10y$ is minimized.

Known: $xy = 250$
$y = \frac{250}{x}$

Objective: minimize $S = 4x + 10y$

Interval:
$x > 0$ & $y > 0$
$\frac{250}{x}$
If $x > 0$, $\frac{250}{x}$ will always be pos.!
My int. is $(0, \infty)$

1-3: $S(x) = 4x + 10\left(\frac{250}{x}\right)$
Minimize $S(x) = 4x + \frac{2500}{x}$ on $(0, \infty)$
$= 4x + 2500x^{-1}$

4: $S'(x) = 4 - 2500x^{-2}$
$= \frac{4}{1} - \frac{2500}{x^2}$
$= \frac{x^2}{x^2} \cdot \frac{4}{1} - \frac{2500}{x^2}$
$= \frac{4x^2 - 2500}{x^2}$

$S'(x) = 0$ when $4x^2 - 2500 = 0$
$\frac{4x^2}{4} = \frac{2500}{4}$
$x^2 = 625$
$\cancel{x = -25}$, $x = 25$
only CV on our interval

$S'(x)$ DNE when $x^2 = 0$
$\cancel{x = 0}$

A cont. fcn on an open int w/ one CV $\Rightarrow$ Use FDT or SDT!
I will use SDT!

$S''(x) = 5000x^{-3} = \frac{5000}{x^3}$
$S''(25) = \frac{5000}{25^3} = +$ $\Rightarrow$ Abs min at $x = 25$

5: $\boxed{x = 25,\ y = \frac{250}{25} = 10}$

5. A company that makes pencil sharpeners has found that it can sell 400 pencil sharpeners each month when the price of each pencil sharpener is \$50. For every one dollar decrease in price, the company can sell 8 additional pencil sharpeners each month.

   (a) Find the price that the company must charge per pencil sharpener to maximize its revenue.

   (b) What is the maximum revenue?

We need to find the price-demand function:

(x, p) — quantity, unit price

$(400, 50)$ $(408, 49)$

$m = \frac{\Delta p}{\Delta x} = \frac{49-50}{408-400} = \frac{-1}{8}$

$p - 50 = -\frac{1}{8}(x - 400)$

$p = -\frac{1}{8}x + 50 + 50$

$p(x) = -\frac{1}{8}x + 100$

Interval

$x \geq 0$ $\quad p \geq 0$

$-\frac{1}{8}x + 100 \geq 0$

$(8)\ 100 \geq \frac{1}{8}x\ (8)$

$800 \geq x$

$[0, 800]$

Revenue = (# of items)(price per item)

$R(x) = x \cdot p(x)$

$R(x) = x\left(-\frac{1}{8}x + 100\right)$

①-③ Maximize $R(x) = -\frac{1}{8}x^2 + 100x$ on $[0, 800]$

④ $R'(x) = -\frac{1}{4}x + 100$, $R'(x) = 0$ when $-\frac{1}{4}x + 100 = 0$

$100 = \frac{1}{4}x$

$400 = x$ — only CV on our int.

Cont. fcn on a closed int ⇒ use C.I.M.

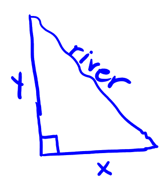
x	R(x)
0	0
400	20000
800	0

⑤ A maximum revenue of \$20000 occurs when they sell 400 pencil sharpeners. The price per pencil sharpener is $p(400) = -\frac{1}{8}(400) + 100 = \$50$.

Ans to part (b)

Ans to part (a)

6. You are building a right-angled triangular garden along a river. The fencing of the left border costs \$4 per foot, while the fencing of the lower border costs \$1 per foot. (No fencing is required along the river.) You want to spend \$200 on the fence. What are the dimensions of the garden that will maximize the amount of area enclosed?



Known

$4y + x = 200$

$\frac{4y}{4} = \frac{200 - x}{4}$

$y = 50 - \frac{1}{4}x$

Objective

Maximize $A = \frac{1}{2}xy$

①-③ Maximize $A(x) = \frac{1}{2}x\left(50 - \frac{1}{4}x\right)$

$A(x) = 25x - \frac{1}{8}x^2$ on $(0, 200)$

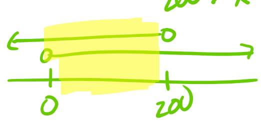
Interval

$x > 0$ $\quad y > 0$

$50 - \frac{1}{4}x > 0$

$(4)\ 50 > \frac{1}{4}x\ (4)$

$200 > x$



$(0, 200)$

④ $A'(x) = 25 - \frac{1}{4}x$ $\quad A'(x) = 0$ when $25 - \frac{1}{4}x = 0$

$25 = \frac{1}{4}x$

$100 = x$ — only CV on our int.

Cont. fcn on an open int w/ one CV ⇒ use FDT or SDT. I will use SDT.

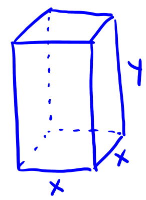
$A''(x) = -\frac{1}{4}$

$A''(100) = -$ ⇒ Abs Max at $x = 100$

$x = 100$ ft $\quad y = 50 - \frac{1}{4}(100) = 25$ ft.

⑤ $x = 100$ ft (lower border)
$y = 25$ ft (left border)
for a max area of $\frac{1}{2}(100)(25) = 1250\text{ ft}^2$

7. A closed rectangular box of volume 324 cubic inches is to be made with a square base. If the material for the bottom costs \$4 per square inch and the material for the sides and top costs \$2 per square inch, find the dimensions of the box that minimize the cost of materials.



Known

$x^2y = 324$

$y = \frac{324}{x^2}$

Objective

minimize $C = 4x^2 + 2(4xy + x^2)$

$C = 4x^2 + 8xy + 2x^2$

$C = 6x^2 + 8xy$

Interval

$x > 0 \quad y > 0$

$\frac{324}{x^2}$ will always be pos. when $x > 0$

$(0, \infty)$

① - ③

$C(x) = 6x^2 + 8x\left(\frac{324}{x^2}\right)$

minimize $C(x) = 6x^2 + 2592x^{-1}$ on $(0, \infty)$

④ $C'(x) = 12x - 2592x^{-2}$

$= 12x - \frac{2592}{x^2}$

$= \frac{12x^3 - 2592}{x^2}$

$C'(x) = 0$ when $12x^3 - 2592 = 0$

$12x^3 = 2592$

$x^3 = 216$

$x = 6$

only CV on our int.

$C'(x)$ DNE when $x^2 = 0$

$x = 0$

Let's use SDT:

$C''(x) = 12 + 5184x^{-3}$

$C''(6) = 12 + \frac{5184}{6^3} = +$ $\Rightarrow$ Abs min at $x = 6$.

⑤

$x = 6''$

$y = \frac{324}{6^2} = 9''$

$\boxed{6'' \times 6'' \times 9'' \text{ for a min cost of } \$648.}$

$C(6) = 6(6)^2 + \frac{2592}{6}$