$$\left[e^{g(x)}\right]' = e^{g(x)}\, g'(x)$$



1. Find the derivative.

(a) $f(x) = \dfrac{e^{\sqrt{x}}}{x+\sqrt{x}}$

$$f'(x) = \frac{(e^{\sqrt{x}})'(x+\sqrt{x}) - (x+\sqrt{x})' e^{\sqrt{x}}}{(x+\sqrt{x})^2} = \frac{e^{\sqrt{x}}(\sqrt{x})'(x+\sqrt{x}) - \left(1+\frac{1}{2\sqrt{x}}\right)e^{\sqrt{x}}}{(x+\sqrt{x})^2}$$

$$= \frac{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}(x+\sqrt{x}) - \left(1+\frac{1}{2\sqrt{x}}\right)e^{\sqrt{x}}}{(x+\sqrt{x})^2} = \frac{\frac{e^{\sqrt{x}}(x+\sqrt{x})}{2\sqrt{x}} - \frac{(2\sqrt{x}+1)}{2\sqrt{x}}e^{\sqrt{x}}}{(x+\sqrt{x})^2}$$

$$= \frac{e^{\sqrt{x}}(x+\sqrt{x}) - (2\sqrt{x}+1)e^{\sqrt{x}}}{2\sqrt{x}(x+\sqrt{x})^2} = \frac{e^{\sqrt{x}}(x+\sqrt{x}-2\sqrt{x}-1)}{2\sqrt{x}(x+\sqrt{x})^2} = \boxed{\frac{e^{\sqrt{x}}(x-\sqrt{x}-1)}{2\sqrt{x}(x+\sqrt{x})^2}}$$

(b) $f(x) = \tan^3(e^{-x} + ex - x^e)$

$$f'(x) = 3\tan^2(e^{-x}+ex-x^e)\left(\tan(e^{-x}+ex-x^e)\right)'$$

power function

$$= 3\tan^2(e^{-x}+ex-x^e)\sec^2(e^{-x}+ex-x^e)(e^{-x}+ex-x^e)'$$

$$= 3\tan^2(e^{-x}+ex-x^e)\sec^2(e^{-x}+ex-x^e)\left(e^{-x}(\cancel{-x})'^{\,-1}+e-ex^{e-1}\right)$$

$$= \boxed{3\tan^2(e^{-x}+ex-x^e)\sec^2(e^{-x}+ex-x^e)(e-e^{-x}-ex^{e-1})}$$

$$\boxed{\begin{array}{l}(\ln x)' = \frac{1}{x}\\ (\log_a x)' = \frac{1}{x\ln a}\end{array}}$$
$a>0,\ a\neq 1$

(c) $f(x) = \log(x^2 - x) = \log_{10}(x^2-x)$

$$f'(x) = \frac{1}{x^2-x}(x^2-x)'\frac{1}{\ln 10} = \boxed{\frac{2x-1}{(x^2-x)\ln 10}}$$

$$\boxed{\begin{array}{l}(e^x)' = e^x\\ (a^x)' = a^x\ln a\end{array}}$$

(d) $f(x) = 3^{\sin x}$

$$f'(x) = \left(3^{\sin x}\right)' = 3^{\sin x}\ln 3\,(\sin x)' = \boxed{3^{\sin x}\ln 3\,(\cos x)}$$

(e) $f(x) = x\sqrt{\ln x}$

$f'(x) = (\cancel{x})'\sqrt{\ln x} + x\left(\sqrt{\ln x}\right)' = \sqrt{\ln x} + (x)\frac{1}{2}[\ln x]^{-1/2}(\ln x)'$

$= \sqrt{\ln x} + \frac{\cancel{x}}{2}(\ln x)^{-1/2}\frac{1}{\cancel{x}}$

$= \boxed{\sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}}$

$[\ln g(x)]' = \frac{g'(x)}{g(x)}$

$[\log_a g(x)]' = \frac{g'(x)}{g(x)\ln a}$

(f) $f(x) = \ln(\ln(3x+1))$

$f'(x) = \frac{1}{\ln(3x+1)}\left(\ln(3x+1)\right)' = \frac{1}{\ln(3x+1)}\frac{1}{3x+1}(\cancel{3x}+1)'$

$= \boxed{\frac{3}{(3x+1)\ln(3x+1)}}$

$\ln\frac{a}{b} = \ln a - \ln b$

(g) $f(x) = \ln\left|\frac{x^2-4}{2x+5}\right| = \ln|x^2-4| - \ln|2x+5|$

$f'(x) = \left(\ln|x^2-4|\right)' - \left(\ln|2x+5|\right)'$

$= \frac{1}{x^2-4}(\cancel{x^2}-4) - \frac{1}{2x+5}(\cancel{2x}+5) = \boxed{\frac{2x}{x^2-4} - \frac{2}{2x+5}}$

$x = e^{\ln x}$

(h) $y = x^{\ln x} = \left(\underbrace{e^{\ln x}}_{x}\right)^{\ln x} = e^{\ln^2 x}$

$y' = \left(e^{\ln^2 x}\right)' = \left(e^{\ln^2 x}\right)(\ln^2 x)' = e^{\ln^2 x}(2\ln x)(\ln x)'$

$= e^{\ln^2 x}(2\ln x)\frac{1}{x} = \frac{2\ln x}{x}e^{\ln^2 x} = \frac{2\ln x}{x}x^{\ln x}$

$= \boxed{2\ln x\left(x^{\ln x - 1}\right)}$

$\sin x = e^{\ln(\sin x)}$

(i) $y = (\sin x)^{\cos x} = e^{\ln(\sin x)\cos x}$

$y' = \left[e^{\ln(\sin x)\cos x}\right]' = e^{\ln(\sin x)\cos x}\left(\ln(\sin x)\cos x\right)'$

$= e^{\ln(\sin x)\cos x}\left[(\ln(\sin x))'\cos x + \ln(\sin x)\left(\cancel{\cos x}\right)'\right]$ $\quad -\sin x$

$= e^{\ln(\sin x)\cos x}\left[\frac{1}{\sin x}\left(\cancel{\sin x}\right)'\cos x - \sin x\,\ln(\sin x)\right]$ $\quad \cos x$

$= e^{\ln(\sin x)\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x\,\ln(\sin x)\right] = \boxed{(\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x\,\ln(\sin x)\right]}$

(j) $y = \arctan(2x+1)$

$$\boxed{\begin{array}{l} (\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \\ (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \\ (\arctan x)' = \frac{1}{1+x^2} \end{array}}$$

$y' = \frac{1}{1+(2x+1)^2}\left(\cancel{2x+1}\right)' = \boxed{\frac{2}{1+(2x+1)^2}}$ $\quad 2$

(k) $y = \sqrt{x}\arcsin(x^3)$

$y' = (\sqrt{x})'\arcsin(x^3) + \sqrt{x}\left(\arcsin(x^3)\right)'$

$= \frac{1}{2}x^{-1/2}\arcsin(x^3) + \sqrt{x}\,\frac{1}{\sqrt{1-(x^3)^2}}\left(\cancel{x^3}\right)'$ $\quad 3x^2$

$= \frac{1}{2\sqrt{x}}\arcsin(x^3) + \frac{3x^2\sqrt{x}}{\sqrt{1-x^6}} = \boxed{\frac{\arcsin(x^3)}{2\sqrt{x}} + \frac{3x^{5/2}}{\sqrt{1-x^6}}}$

(l) $y = (\arccos(4-2x))^5$

$$\boxed{[\arccos(♡)]' = -\frac{1}{\sqrt{1-♡^2}}(♡)'}$$

$y' = 5(\arccos(4-2x))^{5-1}\left(\arccos(4-2x)\right)'$

$= 5(\arccos(4-2x))^4\left[+\frac{1}{\sqrt{1-(4-2x)^2}}\right]\left(\cancel{4-2x}\right)'$ $\quad +2$

$= \boxed{5(\arccos(4-2x))^4\,\frac{2}{\sqrt{1-(4-2x)^2}}}$

2. Use logarithmic differentiation to find the following derivatives.

(a) $\ln f(x) = \ln\left(\frac{x^3+3x}{x^2-4x+1}\right)^{5/2}$

$\ln(x^a) = a\ln x$

$\ln f(x) = \ln\left(\frac{x^3+3x}{x^2-4x+1}\right)^{5/2}$

$= \frac{5}{2}\ln\left(\frac{x^3+3x}{x^2-4x+1}\right)$

$\left[\ln f(x)\right]' = \left[\frac{5}{2}\left[\ln(x^3+3x) - \ln(x^2-4x+1)\right]\right]'$

$\frac{f'(x)}{f(x)} = \frac{5}{2}\left[\frac{1}{x^3+3x}\underbrace{(x^3+3x)'}_{3x^2+3} - \frac{1}{x^2-4x+1}\underbrace{(x^2-4x+1)'}_{2x-4}\right]$

$f(x)\frac{f'(x)}{f(x)} = \frac{5}{2}\left[\frac{3x^2+3}{x^3+3x} - \frac{2x-4}{x^2-4x+1}\right]f(x)$

$\boxed{f'(x) = \frac{5}{2}\left(\frac{x^3+3x}{x^2-4x+1}\right)^{5/2}\left[\frac{3x^2+3}{x^3+3x} - \frac{2x-4}{x^2-4x+1}\right]}$

(b) $\ln y = \ln\frac{(x+1)^{151}(5-\sin x)^3}{(3x-7)^{2024}}$

$\ln(ab) = \ln a + \ln b$

$\ln y = \ln\frac{(x+1)^{151}(5-\sin x)^3}{(3x-7)^{2024}}$

$\ln y = \ln(x+1)^{151} + \ln(5-\sin x)^3 - \ln(3x-7)^{2024}$

$[\ln y]' = [151\ln(x+1) + 3\ln(5-\sin x) - 2024\ln(3x-7)]'$

$\frac{y'}{y} = \frac{151}{x+1} + \frac{3}{5-\sin x}\underbrace{(5-\sin x)'}_{-\cos x} - \frac{2024}{3x-7}\underbrace{(3x-7)'}_{3}$

$y\frac{y'}{y} = \left[\frac{151}{x+1} - \frac{3\cos x}{5-\sin x} - \frac{6072}{3x-7}\right]y$

$\boxed{y' = \frac{(x+1)^{151}(5-\sin x)^3}{(3x-7)^{2024}}\left[\frac{151}{x+1} - \frac{3\cos x}{5-\sin x} - \frac{6072}{3x-7}\right]}$

(c) $\ln y = \ln\frac{e^x\sqrt{x^2+2}}{\sqrt[3]{x}}$

$\ln y = \ln\frac{e^x\sqrt{x^2+2}}{\sqrt[3]{x}}$

$= \ln e^{x} + \ln(x^2+2)^{1/2} - \ln x^{1/3}$

$[\ln y]' = \left[x + \frac{1}{2}\ln(x^2+2) - \frac{1}{3}\ln x\right]'$

$\frac{y'}{y} = 1 + \frac{1}{2}\frac{1}{x^2+2}(x^2+2)' - \frac{1}{3}\frac{1}{x}$

$\frac{y'}{y} = 1 + \frac{1}{\cancel{2}}\frac{1}{x^2+2}(\cancel{2}x) - \frac{1}{3x}$

$y\frac{y'}{y} = \left[1 + \frac{x}{x^2+2} - \frac{1}{3x}\right]y$

$\boxed{y' = \frac{e^x\sqrt{x^2+2}}{\sqrt[3]{x}}\left[1 + \frac{x}{x^2+2} - \frac{1}{3x}\right]}$

3. Find $f''(x)$ for the function $f(x) = (1 + x^2)\tan x$.

first derivative

$$f'(x) = (1+x^2)'\tan x + (1+x^2)(\tan x)'$$

$$f'(x) = 2x\tan x + (1+x^2)\sec^2 x$$

$$f''(x) = (2x)'\tan x + 2x(\tan x)' + (1+x^2)'\sec^2 x + (1+x^2)(\sec^2 x)'$$

$$= 2\tan x + \underline{2x(\sec^2 x)} + \underline{2x(\sec^2 x)} + (1+x^2)\, 2\sec x(\sec x)'$$

$$= 2\tan x + 4x(\sec^2 x) + 2(1+x^2)\underbrace{(\sec x)(\sec x}_{\sec^2 x}\, \tan x)$$

$$\boxed{f''(x) = 2\tan x + 4x\sec^2 x + 2(1+x^2)\sec^2 x \tan x}$$

4. Find $f^{(58)}(x)$ if $f(x) = e^{-2x} + \cos(3x)$.

$y = e^{-2x}$

$y' = (-2)e^{-2x}$

$y'' = (-2)(-2)e^{-2x} = (-2)^2 e^{-2x}$

$y''' = (-2)^2 (e^{-2x})'$

$y''' = (-2)^3 e^{-2x}$

$y^{(58)} = (-2)^{58} e^{-2x}$ , $2^{58}$

58 is even , $(-2)^{58} = 2^{58}$

$\left[e^{-2x}\right]^{(58)} = 2^{58} e^{-2x}$

$y = \cos 3x$

$y' = -3\sin 3x = 3(-\sin 3x)$

$y'' = -9\cos 3x = 3^2(-\cos 3x)$

$y''' = 27\sin 3x = 3^3(\sin 3x)$

$y^{IV} = 81\cos 3x = 3^4(\cos 3x)$

$y^{V} = 3^5(-\sin 3x)$

- - - - . .

$58 = 56 + 2$

$y^{(58)} = 3^{58}(-\cos 3x)$

$$\boxed{f^{(58)}(x) = 2^{58} e^{-2x} - 3^{58}\cos 3x}$$

$\vec{r}(t) = \langle x(t), y(t) \rangle$ $\qquad$ $\vec{r}\,''(t) = \langle x''(t), y''(t) \rangle$

$\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$

5. The vector function $\mathbf{r}(t) = < t + e^{4t}, -t\cos(2t) >$, $0 \le t \le 2\pi$, represents the position of a particle at time $t$. Find the velocity and acceleration vectors of the object at $t = \frac{\pi}{4}$.

velocity $\vec{v}(t) = \vec{r}\,'(t) = \langle (t+e^{4t})', (-t\cos 2t)' \rangle$

$= \langle 1 + 4e^{4t}, (-t)'\cos 2t - t(\cos 2t)' \rangle$ $\quad (-t)' = -1$, $(\cos 2t)' = -2\sin 2t$

$\boxed{\vec{v}(t) = \langle 1+4e^{4t}, -\cos 2t + 2t\sin 2t \rangle}$

$\vec{v}(\frac{\pi}{4}) = \langle 1+4e^{\frac{4\pi}{4}}, -\cos\frac{2\pi}{4} + 2\frac{\pi}{4}\sin\frac{2\pi}{4} \rangle$

$\vec{v}(\frac{\pi}{4}) = \langle 1+4e^{\pi}, -\cos\frac{\pi}{2} + \frac{\pi}{2}\sin\frac{\pi}{2} \rangle$ $\quad \cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$

$\boxed{\vec{v}(\frac{\pi}{4}) = \langle 1+4e^{\pi}, \frac{\pi}{2} \rangle}$

acceleration $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \langle (1+4e^{4t})', (-\cos 2t + 2t\sin 2t)' \rangle$

$= \langle 16e^{4t}, 2\sin 2t + 2(t)'\sin 2t + 2t(\sin 2t)' \rangle$ $\quad (t)' = 1$, $(\sin 2t)' = 2\cos 2t$

$= \langle 16e^{4t}, 2\sin 2t + 2\sin 2t + 4t\cos 2t \rangle$

$\boxed{\vec{a}(t) = \langle 16e^{4t}, 4\sin 2t + 4t\cos 2t \rangle}$

$\vec{a}(\frac{\pi}{4}) = \langle 16e^{4\frac{\pi}{4}}, 4\sin\frac{2\pi}{4} + 4\frac{\pi}{4}\cos\frac{2\pi}{4} \rangle$

$= \langle 16e^{\pi}, 4\sin\frac{\pi}{2} + \pi\cos\frac{\pi}{2} \rangle$ $\quad \sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$

$\boxed{\vec{a}(\frac{\pi}{4}) = \langle 16e^{\pi}, 4 \rangle}$

6. Consider the curve $x = t^2 - 10t - 3$, $y = 5t^2 + t$.

(a) Find the equation of the tangent line at the point (8,4).

(b) At what point(s) is the tangent line to the graph parallel to the line $7x + 2y = 19$.

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(5t^2+t)'}{(t^2-10t-3)'} = \frac{10t+1}{2t-10}$ - slope of the tangent line

(a) slope @ (8,4).
Find the value of $t$ such that: $\begin{cases} t^2-10t-3=8 \longrightarrow t^2-10t-11=0 \\ 5t^2+t=4 \end{cases}$

$(t-11)(t+1)=0$

$\boxed{t_1=11}$, $\boxed{t_2=-1}$

$5t^2+t=4$ plug in $t_1=11$, $t_2=-1$

not valid $\cancel{t=11}$: $5(11)^2+11 \overset{?}{\neq} 4$

$t=-1$: $5(-1)^2-1 = 5-1 = 4$

The point (8,4) corresponds to $t=-1$

slope @ $(t=-1)$ is $\left.\frac{dy}{dx}\right|_{t=-1} = \frac{10(-1)+1}{2(-1)-10} = \frac{-9}{-12} = \frac{3}{4}$

tangent line @ (8,4) is $\boxed{y-4 = \frac{3}{4}(x-8)}$

(b) $7x+2y=19 \Longrightarrow 2y = 19-7x$

$y = -\frac{7}{2}x + \frac{19}{2} \longleftarrow$ slope is $-\frac{7}{2}$

Find $t$ such that $\frac{10t+1}{2t-10} = -\frac{7}{2}$

$2(10t+1) = -7(2t-10)$

$20t+2 = -14t+70$

$34t = 68 \Longrightarrow \boxed{t=2}$

The point on the curve corresponding to $t=2$ is

$x(2) = 2^2-10(2)-3 = 4-20-3 = -19$

$y(2) = 5(2^2)+2 = 22$

$\boxed{(-19, 22)}$

8. Find the point(s) on the curve $x = 1 - 2\cos t$, $y = 2 + 3\sin t$ where the tangent is horizontal or vertical.

9. Find the vector and parametric equations for the line tangent to the curve $\mathbf{r}(t) = < \overbrace{1-4t}^{x}, \overbrace{2t-3t^2}^{y} >$ at the point $P(-11,-21)$.

Find $t$ such that $\vec{r}(t) = < 1-4t, 2t-3t^2 > = < -11, -21 >$

or $\begin{cases} 1-4t = -11 \Rightarrow 4t = 12 \Rightarrow \boxed{t=3} \\ 2t-3t^2 = -21 \end{cases}$

Plug $t=3$ into $2t-3t^2 = -21$

$2(3)-3(3)^2 \overset{?}{=} -21$

$6 - 27 \overset{?}{=} -21$

$-21 = -21$

$\vec{r}(3) = < -11, -21 >$

slope of the tangent line is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2-6t}{-4} = -\frac{1-3t}{2} = \frac{3t-1}{2}$

slope @ $t=3$ is $\left.\frac{dy}{dx}\right|_{t=3} = \frac{3(3)-1}{2} = \frac{9-1}{2} = 4$

tangent line: $\boxed{y+21 = 4(x+11)}$

10. The ball is tossed into the air. Its position at time $t$ is given by $\mathbf{r}(t) =< 5t, \underbrace{100t - 16t^2}_{Y(t)} >$.

(a) Find the velocity and the speed of the ball when $t = 2$.

(b) How high does the ball go?

(c) With what speed does the ball hit the ground?

(a) $\vec{v}(t) = \vec{r}\,'(t) = < 5, 100 - 32t >$

velocity $\vec{v}(2) = < 5, 100 - 64 >$

$\boxed{\vec{v}(2) = < 5, 36 >}$ velocity

speed $s(2) = |\vec{v}(2)| = \sqrt{5^2 + 36^2} = \sqrt{1321}$

(b) in the heighest point , $y'(t) = 0$

$y(t) = 100t - 16t^2$

11. If $\mathbf{r}(t) =< t^3, t^2 >$ represents the position of a particle at time $t$, find the angle between the velocity and the acceleration vector at time $t = 1$.

12. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after 5 sec.

13. If a ball is thrown vertically upward with a velocity of 144 ft/s, then its height after $t$ seconds is $s = 144t - 16t^2$.

    (a) What is the maximum height reached by the ball?
    (b) What is the velocity of the ball when it is 320 ft above the ground on his way up?
    (c) What is the velocity of the ball when it is 320 ft above the ground on his way down?
    (d) When will the ball hit the ground?
    (e) With what velocity does the ball hit the ground?

7. At what point(s) does the curve parametrized by $x = t^2 - 6t + 5$, $y = t^2 + 4t + 3$ have a horizontal or vertical tangent?

slope of a tangent is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

vertical tangent $\Rightarrow$ slope is $\frac{\pi}{2}$ or $\boxed{x'(t) = 0}$

$x(t) = t^2 - 6t + 5$

$x'(t) = 2t - 6 = 0$

$\boxed{\text{vertical tangent @ } (-4, 24)}$ when $\boxed{t = 3}$

$x(3) = 3^2 - 3(6) + 5 = -4$

$y(3) = 3^2 + 4(3) + 3 = 24$

horizontal tangent is when the slope is zero or $\boxed{y'(t) = 0}$

$y(t) = t^2 + 4t + 3$

$y'(t) = 2t + 4 = 0 \Rightarrow \boxed{t = -2}$

$x(-2) = 2^2 + 6(-2) + 5 = 4 + 12 + 5 = 21$

$y(-2) = 2^2 + 4(-2) + 3 =$