1 Week 15 HOGU: 6.2, Final Exam Review Part 2

Problem 1. The John Weeks Enterprise is booming! The business needs to save up $\$150,000$ for a planned expansion. They make an initial deposit of $\$25,000$ and plan on depositing $\$500$ at the end of each month in T-bills, which currently yield 5.16% APR. Say that interest in these T-bills is compounded monthly. How many years will it take the John Weeks Enterprise to save up the $\$150,000$ they need?

TVM Solver

N = ? → 147.7138...

I% = 5.16

PV = -25000

PMT = -500

FV = 150000

P/Y = 12

C/Y = 12

PMT: END BEGIN

147.71 months → about $12\frac{1}{4}$ years

$\left(\frac{147.71}{12} \approx 12.309...\right)$

Problem 2. You are 18 years of age and want to be ready to retire at age 65! You put $\$1,000$ in a stable mutual fund that has a 3% APR. You continue investing $\$100$ ~~in~~ 300 that mutual fund every month from now until you turn 65. How much money do you have to retire on at age 65?

To supplement your employer's retirement fund, exactly

Interest is compounded monthly

N = (65 - 18) * 12

I% = 3

PV = -1000

PMT = -300

FV = ? → $374,739.31

P/Y = 12

C/Y = 12

PMT: END BEGIN

Problem 3. You are building the boat "Sailing for Weeks", which will currently cost \$44,750 to craft. You make a down payment of \$10,000 and finance the rest with a 10-year loan. The loan charges 7% interest, compounded quarterly.

(a) After making quarterly payments on the boat for 3 years, how much of the principal of your loan is still unpaid?

TVM Solver, Twice!

N = 10 · 4
I% = 7
PV = 44750-10000
PMT = ? → -\$1215.28 quarterly payment
FV = 0
P/Y = 4
C/Y = 4
PMT: END BEGIN

N = 3 · 4
I% = 7
PV = 44750-10000
PMT = -1215.280164...
FV = ? → \$26720.31
P/Y = 4
C/Y = 4
PMT: END BEGIN

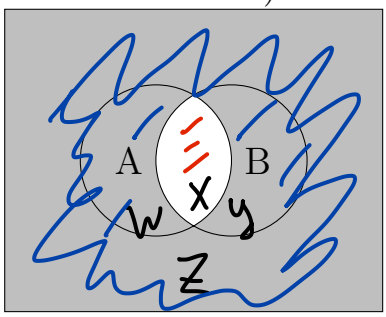
(b) After 3 years on your old loan above, you find a loan company that will refinance your loan at an APR of 5%, compounded quarterly, with a payment schedule that has you paying off the loan in 5 years! You decide to refinance your purchase and to make quarterly payments on the new loan. How much will your quarterly payments be?

N = 5 · 4
I% = 5
PV = 26720.31
PMT = ?

FV = 0
P/Y = 4
C/Y = 4
PMT: END BEGIN

↳ -\$1518.26 quarterly payment on new loan

Problem 4. Using union, intersection, and complements, how would you describe the shaded-in regions of these Venn diagram? (There is more than one correct answer.)



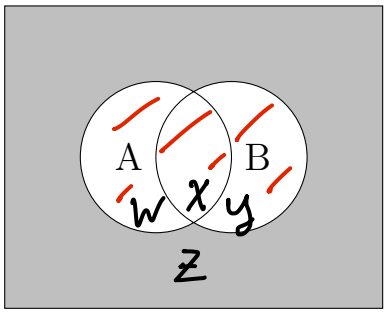
Easier to describe the NOT shaded in region

NOT shaded in: $\{x\}$

$A=\{w,x\}$ $B=\{x,y\}\rightarrow A\cap B=\{x\}$

Shaded in: $\{w,y,z\}$

$A\cap B=\{x\}\Rightarrow \boxed{(A\cap B)^c}=\{w,y,z\}$



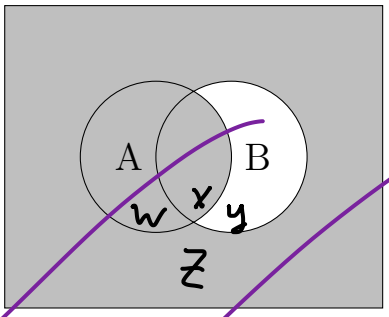
Easier to describe the NOT shaded in region

NOT shaded in: $\{w,x,y\}$

$A=\{w,x\}$ $B=\{x,y\}$ $A\cup B=\{w,x,y\}$

Shaded in: $\{z\}$

$A\cup B=\{w,x,y\}\Rightarrow \boxed{(A\cup B)^c}=\{z\}$



Problem 5. The following is a probability distribution with a missing entry:

Outcome	1	2	3	4	5
Probability	$\frac{18}{100}$	$\frac{23}{100}$	X	$\frac{9}{100}$	$\frac{33}{100}$

Let $A = \{1, 3, 5\}$ and let $B = \{2, 4, 5\}$. What is $P\left(A \cup B^C\right)$?

$A = \{1,3,5\}$ $B = \{2,4,5\} \longrightarrow B^c = \{1,3\}$

$A \cup B^c = \{1,3,5\} \cup \{1,3\} = \{1,3,5\}$

$\frac{18}{100} + \frac{23}{100} + X + \frac{9}{100} + \frac{33}{100} = 1 \longrightarrow \frac{83}{100} + X = \frac{100}{100}$

all probabilities add to 1 $\frac{100}{100}$

$-\frac{83}{100}$ $-\frac{83}{100}$

$X = \frac{100-83}{100} = \frac{17}{100}$

$P(A \cup B^c) = P(\{1,3,5\}) = \frac{18}{100} + \frac{17}{100} + \frac{33}{100} = \boxed{\frac{68}{100}}$

Problem 6. Given that F and G are two events of an experiment with $P(F) = 0.4$, $P(G) = 0.5$, and $P(F \cap G) = 0.2$, calculate the following probabilities:

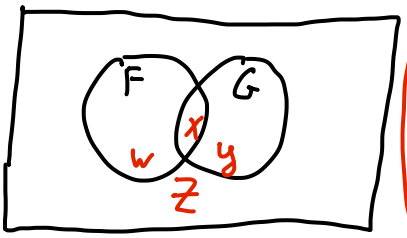
(a) $P(F^C)$

Complement Rule

$$P(F^c) = 1 - P(F) = 1 - 0.4 = \boxed{0.6}$$

(b) ~~$P(G^C)$~~ Now say $P(F) = 0.4$, $P(G) = 0.5$, and $P(F \cup G^c) = 0.8$.

What is $P(F \cap G)$?



$w + x = 0.4$

$x + y = 0.5$

$w + x + z = 0.8$

$w + x + y + z = 1$

Goal: Solve for x

$= F \cap G$

① Subtract

$w + x + z = 0.8$

$-(w + x = 0.4)$

$z = 0.4$

$w + x = 0.4$

$x + y = 0.5$

$w + x + y = 1 - z = 1 - 0.4 = 0.6$

② Subtract

$w + x + y = 0.6$

$-(x + y = 0.5)$

$w = 0.1$

③ $(0.1) + x = 0.4$

$x = 0.3$

(b) ~~(c)~~ $P(F \cup G)$

Union Rule

$$P(F \cup G) = P(F) + P(G) - P(F \cap G)$$
$$= 0.4 + 0.5 - 0.2$$
$$= \boxed{0.7}$$

Problem 7. A local group is sponsoring a game at the Renaissance Fair! A foolish jester asks you to pay \$1 to play the game, then flips a fair two-sided coin. As it is in the air, the player calls "heads" or "tails". If the coin lands on the side the player called out, they win \$5! Otherwise, the player wins nothing.

(a) Write a probability distribution calculating the *net winnings* that the player has from playing a game.

Let X be the amount of net winnings, in dollars, that a player makes from playing this game.

^of X

X	-1	$-1+5$
$P(X)$	0.5	0.5

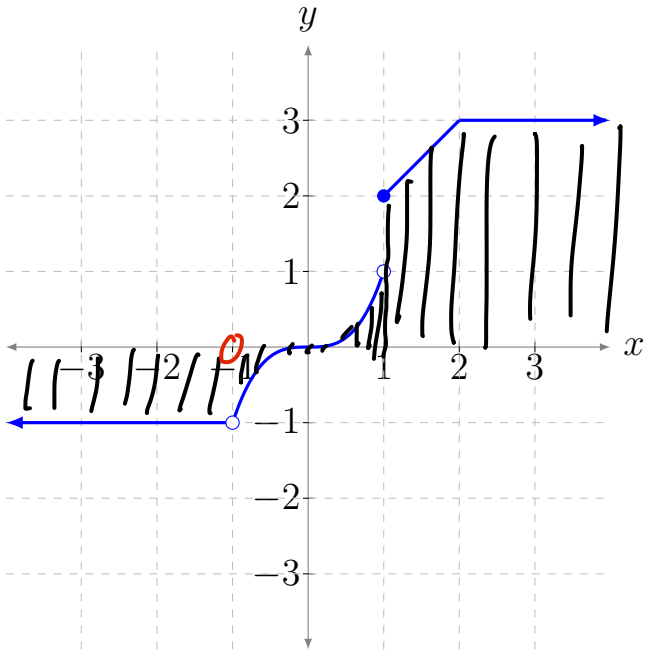
(b) What is the expected amount of net winnings for the player? Is this a fair game?

$$E(X) = -1 \cdot (0.5) + 4 \cdot (0.5) = 1.5$$

The player is expected to earn \$1.50 each time they play this game.

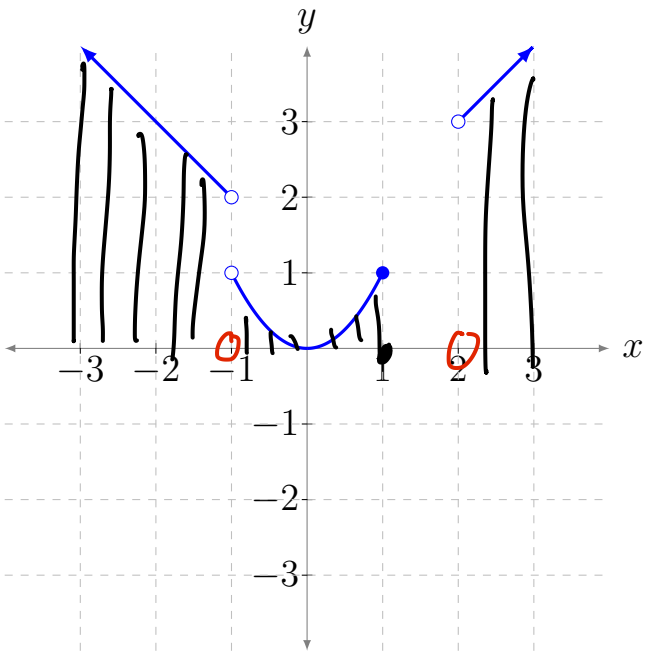
This is NOT a fair game because the expected net winnings is not equal to \$0.

Problem 8. Find the domain and range of the following functions.



Domain: $(-\infty, -1) \cup (-1, \infty)$

Range: $[-1, 1) \cup [2, 3]$



Domain: $(-\infty, -1) \cup (-1, 1] \cup (2, \infty)$

Range: $[0, 1] \cup (2, \infty)$

Problem 9. Compute the domain of the following functions. Then state all holes and vertical asymptotes of the function. (Remember that, for holes, you must give an x- and y-coordinate. Write your answer as (x, y).)

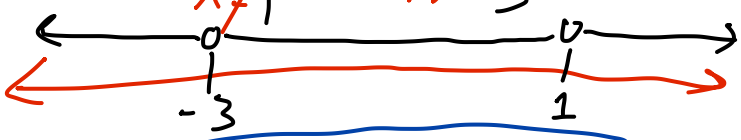
(a) $f(x) = \dfrac{(x+3)^2}{(x-1)^2(x+3)}$

Denominator $\neq 0$!

$(x-1)^2(x+3) \neq 0$

$x-1 \neq 0$ & $x+3 \neq 0$

$x \neq 1$ $\quad$ $x \neq -3$



$(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

$x=1$ is a vertical asymptote!

No factors $x-1$ cancel in denominator

$x=-3$ is the x-coordinate of a hole!

All factors $x+3$ cancel in denominator

$f(x) = \dfrac{x+3}{(x-1)^2} \rightarrow f(-3) = \dfrac{-3+3}{(-3-1)^2}$

$= \dfrac{0}{16} = 0$

hole at $(-3, 0)$

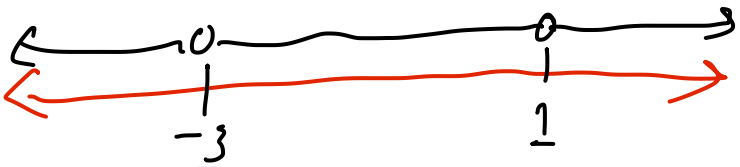
(b) $f(x) = \dfrac{(x+3)}{(x-1)^2(x+3)^2}$

Denominator $\neq 0$!

$(x-1)^2(x+3)^2 \neq 0$

$x-1 \neq 0$ $\quad$ $x+3 \neq 0$

$x \neq 1$ $\quad$ $x \neq -3$



$(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

$x=1$ is a vertical asymptote!

No factors $x-1$ cancel in denominator

$x=-3$ is a vertical asymptote!

Not all factors $x+3$ cancel in denominator

Problem 10. Compute and completely simplify the difference quotient for the function $f(x) = \sqrt{x}$.

(a) What is $f(x+h) - f(x)$?

$f(x+h) = \sqrt{x+h}$

$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x}$

(b) Write your answer from above over 1, like $\frac{f(x+h)-f(x)}{1}$, then *rationalize the numerator*. Simplify your answer.

$$\frac{\sqrt{x+h}-\sqrt{x}}{1} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{\sqrt{x+h}+\sqrt{x}}$$

$$\overset{\text{FOIL}}{=} \frac{\sqrt{x+h}\sqrt{x+h} + \sqrt{x+h}\cdot\sqrt{x} - \sqrt{x}\cdot\sqrt{x+h} - \sqrt{x}\cdot\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{(x+h)-(x)}{\sqrt{x+h}+\sqrt{x}} = \frac{h}{\sqrt{x+h}+\sqrt{x}}$$

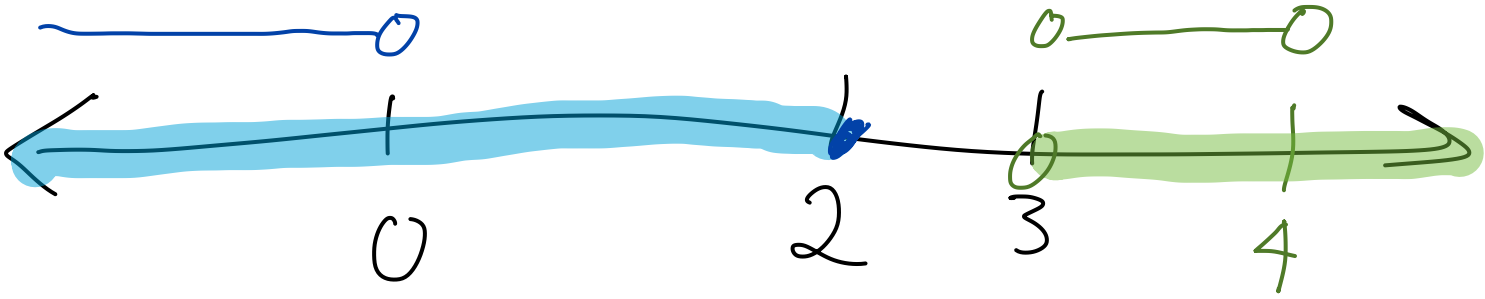
(c) Using your answers from above, calculate $\frac{f(x+h)-f(x)}{h}$, the difference quotient of $f(x)=\sqrt{x}$.

$$\frac{\frac{h}{\sqrt{x+h}+\sqrt{x}}}{h} = \frac{h}{\sqrt{x+h}+\sqrt{x}} \cdot \frac{1}{h} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Problem 11. Find the domain of the following function:

$$f(x) = \begin{cases} \ln(x) & \text{if } x \leq 2 \\ \dfrac{9}{\sqrt{x-1}} & \text{if } x > 3 \end{cases}$$

$\begin{cases} \ln(-x) & \text{if } x \leq 2 \\ \dfrac{9}{\sqrt{4-x}} & \text{if } x > 3 \end{cases}$



Logarithm $\ln(\star)$: $\star > 0$!

$-x > 0$

$x < 0$

$\frac{9}{4-x}$: Denominator $\neq 0$!

$\sqrt{4-x} \neq 0 \to 4-x \neq 0 \to x \neq 4$

Even root $\sqrt{\star}$: $\star \geq 0$!

$4 - x \geq 0 \to x \leq 4$

Together: $x < 4$

$$(-\infty, 0) \cup (3, 4)$$

Problem 12. Solve the equation $e^{4x} = 5e^{7x}$ for x.

$e^{4x} = 5e^{7x}$ Isolate x on one side

$\frac{e^{4x}}{5e^{4x}} \quad \frac{5e^{7x}}{5e^{4x}}$

$\frac{1}{5} = e^{7x \cdot 4x} = e^{3x}$ x is in exponent; taking logs can bring it down!

$\ln\left(\frac{1}{5}\right) = \ln\left(e^{3x}\right)$

$\ln\left(\frac{1}{5}\right) = 3x \ln(e) = 3x$

$x = \boxed{\frac{\ln\left(\frac{1}{5}\right)}{3}}$ OR $\frac{\ln(1) - \ln(5)}{3} = \boxed{\frac{-\ln(5)}{3}}$

Problem 13. Write the expression $\frac{1}{3}\ln(x) - \ln(x+y) + 4\ln(2z)$ as a single logarithm. Assume all variables represent positive numbers.

$\frac{1}{3}\ln(x) = \ln\left(x^{\frac{1}{3}}\right) = \ln\left(\sqrt[3]{x}\right)$

$4\ln(2z) = \ln\left((2z)^4\right) = \ln\left(16z^4\right)$

$\frac{1}{3}\ln(x) - \ln(x+y) + 4\ln(2z) = \ln\left(\sqrt[3]{x}\right) - \ln(x+y) + \ln\left(16z^4\right)$

Positive terms go in numerator: multiply them!

Negative terms go in denominator: divide them!

$= \boxed{\ln\left(\frac{16z^4 \cdot \sqrt[3]{x}}{x+y}\right)}$