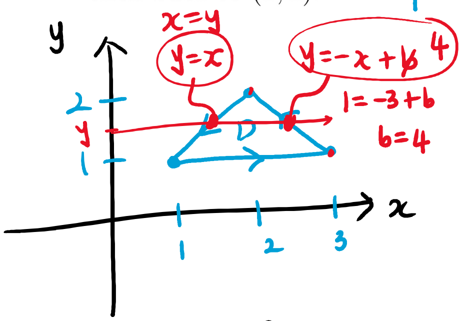
24A_WIR_M251_H9_solutions

Monday, April 22, 2024 6:06 PM

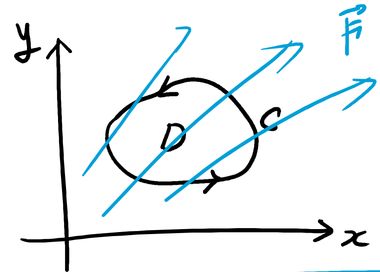
24A_WIR_
M251_H9...

Note ♯9: Sections 16.4-16.5

Problem 1. Evaluate $\oint_C \underbrace{y^2}_{P}\,dx + \underbrace{x}_{Q}\,dy$ where C is the triangular path from $(1,1)$ to $(3,1)$ to $(2,2)$ then back to $(1,1)$.



$$\left.\begin{aligned} Q_x &= 1 \\ P_y &= 2y \end{aligned}\right\} \Rightarrow Q_x - P_y = 1-2y$$



$$\int_C y^2dx + xdy = \iint_D 1-2y\,dA = \int_1^2\int_y^{4-y} 1-2y\,dxdy$$

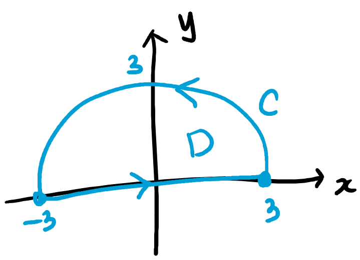
$$= \cdots \boxed{}$$

$\vec{F} = \langle P, Q\rangle$ $\quad C: \vec{r}(t) = \langle x(t), y(t)\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C Pdx + Qdy = \iint_D Q_x - P_y\,dA$$

Green's thm

Problem 2. Evaluate $\oint_C \underbrace{\left(\cos\left(e^x\right) + y^2\right)}_{P}dx + \underbrace{\left(x^2 + \sqrt{y^3+y}\right)}_{Q}dy$, where C consists of the line segment from $(-3,0)$ to $(3,0)$ and the top half of $x^2+y^2=9$. Assume counterclockwise orientation.

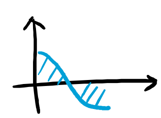


$$\left.\begin{aligned} Q_x &= 2x \\ P_y &= 2y \end{aligned}\right\} \quad Q_x - P_y = 2x - 2y$$

$$= \iint_D 2x-2y\,dA = \int_0^{\pi}\int_0^3 \underbrace{\left(2(r\cos\theta) - 2(r\sin\theta)\right) r}_{2r^2(\cos\theta - \sin\theta)}\,drd\theta$$

$$= \left(\int_0^3 2r^2dr\right)\left(\int_0^{\pi} \cancel{\cos\theta} - \sin\theta\,d\theta\right)$$

integral is 0 by symmetry



$$= \left[\tfrac{2}{3}r^3\right]_0^3 \left[\cos\theta\right]_0^{\pi}$$

$$= (18)(-2) = \boxed{-36}$$

Problem 3. Evaluate $\oint_C y^2 dx + x^2 dy$ where C is the boundary of the region bounded by the semicircle $y = \sqrt{4 - x^2}$ and the x axis. Assume positive orientation.

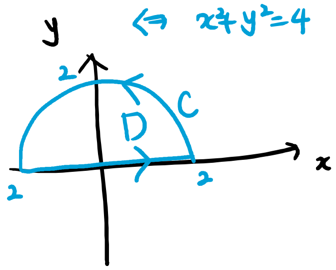
P Q

$0 \le$

Counterclockwise

$y^2 = 4 - x^2$

$\Leftrightarrow x^2 + y^2 = 4$



$$Q_x - P_y = 2x - 2y$$

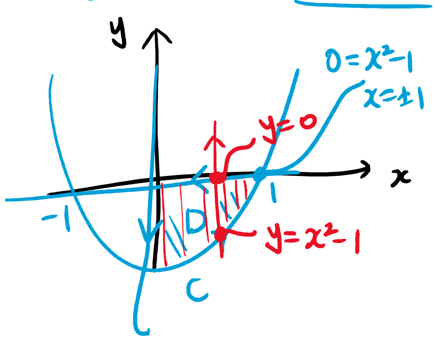
$$\iint_D 2x - 2y \, dA = \int_0^{\pi}\int_0^{2} (2r\cos\theta - 2r\sin\theta)\, r\,dr\,d\theta$$

$$= \cdots \boxed{}$$

Problem 4. Evaluate $\oint_C (y^2 + x^{10})dx + y^7 dy$, where C is the curve that encloses the region bounded by $x = 0$, $y = 0$, and $y = x^2 - 1$, traversed counterclockwise.

P Q

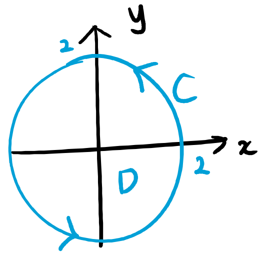
in the fourth quadrant



$$Q_x - P_y = 0 - 2y = -2y$$

$$\iint_D -2y \, dA = \int_0^1 \int_{x^2-1}^{0} -2y \, dy\,dx = \cdots \boxed{}$$

Problem 5. Suppose a particle travels one revolution counterclockwise around the circle $x^2+y^2=4$ under the force field $\mathbf{F}(x,y) = \langle \underbrace{y^3+\sin(x)}_{P}, \underbrace{e^y - x^3}_{Q} \rangle$. Find the work done by $\mathbf{F}$.



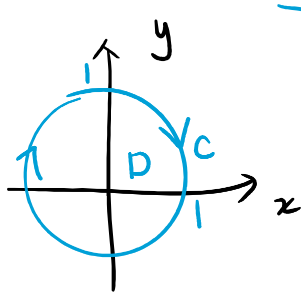
$$W = \int_C \vec{F}\cdot d\vec{r} = \iint_D -3x^2-3y^2 \, dA$$

$$\left(Q_x - P_y = -3x^2 - 3y^2 \right)$$

$$= \int_0^{2\pi}\int_0^2 -3\, r^2 \cdot r\,dr\,d\theta = \int_0^2 -3\, r^3\, dr \int_0^{2\pi} d\theta$$

$$= -3\left[\frac{r^4}{4}\right]_0^2 \cdot 2\pi = -3(4)(2\pi) = \boxed{-24\pi}$$

Problem 6. Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x,y) = \langle \underbrace{e^x - y^3}_{P}, \underbrace{\cos(y)+x^3}_{Q} \rangle$. Find the work done by $\mathbf{F}$.



$$W = \int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r} = -\iint_D 3(x^2+y^2)\, dA$$

$$\left(Q_x - P_y = 3x^2 + 3y^2 \right)$$

$$= -\int_0^{2\pi}\int_0^1 3\, r^2 \cdot r\,dr\,d\theta$$

$$= -\int_0^1 3\, r^3\, dr \int_0^{2\pi} d\theta$$

$$= -\frac{3}{4}\left[r^4\right]_0^1 \cdot 2\pi = -\frac{3}{4}\cdot 2\pi = \boxed{-\frac{3}{2}\pi}$$

Problem 7. Find the divergence and curl of $\mathbf{F}(x,y,z) = xe^{yz}\mathbf{j} + x^3z\mathbf{k}$.

Define $\left(\frac{\partial}{\partial x}\right) f = \frac{\partial f}{\partial x}$.

gradient $\nabla f = \langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f \rangle$

$= \boxed{\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle} f$

$=: \nabla$ (del)

divergence $\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle$

$= P_x + Q_y + R_z$

curl $\nabla \times \vec{F} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$

$= (R_y - Q_z)\hat{i} - (R_x - P_z)\hat{j} + (Q_x - P_y)\hat{k}$

① $\nabla \cdot \vec{F} = 0_x + (xe^{yz})_y + (x^3 z)_z$

$= \boxed{xze^{yz} + x^3}$

② $\nabla \times \vec{F} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & xe^{yz} & x^3 z \end{vmatrix}$

$= \boxed{-xye^{yz}\hat{i} - 3x^2 z\hat{j} + e^{yz}\hat{k}}$

Problem 8. Find the divergence and curl of $\mathbf{F}(x,y,z) = \langle xyz, 2\sin(xz), y^2z^3 \rangle$.

$\frac{\partial}{\partial x} = \partial_x$

① divergence : $\boxed{\operatorname{div}\vec{F} = \nabla \cdot \vec{F}} = (xyz)_x + (2\sin(xz))_y + (y^2z^3)_z$

$= yz + 0 + 3y^2z^2 = \boxed{yz + 3y^2z^2}$

② Curl : $\boxed{\operatorname{curl}\vec{F} = \nabla \times \vec{F}} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ \partial_x & \partial_y & \partial_z \\ xyz & 2\sin(xz) & y^2z^3 \end{vmatrix}$

$= (2yz^3 - 2\cos(xz)\cdot x)\hat{i} - (0 - xy)\hat{j} + (2\cos(xz)z - xz)\hat{k}$

$= \boxed{(2yz^3 - 2x\cos(xz))\hat{i} + xy\hat{j} + (2z\cos(xz) - xz)\hat{k}}$

Problem 9. Let f be a scalar field and $\mathbf{F}$ a vector field. Describe if each expression is a scalar field, vector field, or not meaningful.

(a) curl f $= \underbrace{\nabla}_{3d} \times \underbrace{f}_{1d}$ $\Rightarrow$ not meaningful.

(b) curl(grad f) $= \underbrace{\nabla}_{3d} \times \underbrace{(\underbrace{\nabla}_{3d} \underbrace{f}_{1d})}_{3d}$ $\Rightarrow$ vector field.

(c) grad(div $\mathbf{F}$) $= \underbrace{\underbrace{\nabla}_{3d}\underbrace{(\underbrace{\nabla}_{3d} \cdot \underbrace{\vec{F}}_{3d})}_{1d}}_{3d}$ $\Rightarrow$ Vector field

Problem 10. Is $\mathbf{F}(x,y,z) = \langle 3x^2y + 4y^2z, x^3 + 8xyz, 4xy^2 - e^z \rangle$ a conservative vector field? If so, find a potential function for $\mathbf{F}$.

① $\nabla \times \vec{F} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3x^2y+4y^2z & x^3+8xyz & 4xy^2-e^z \end{vmatrix}$

$\langle P,Q \rangle = \boxed{\vec{F} = \nabla f} = \langle f_x, f_y \rangle \Leftrightarrow \boxed{Q_x - P_y = 0}$

$\langle P,Q,R \rangle = \boxed{\vec{F} = \nabla f} = \langle f_x, f_y, f_z \rangle \Leftrightarrow \boxed{\text{Curl } \vec{F} = \vec{0}}$

$= \cancel{(8xy - 8xy)}\,\hat{i} - \cancel{(4y^2 - 4y^2)}\,\hat{j} + \cancel{((3x^2+8yz) - (3x^2+8yz))}\,\hat{k} = \vec{0}$

$\Rightarrow \vec{F}$ is conservative!

② $3x^2y + 4y^2z = f_x \Rightarrow \int f_x\,dx = f = \boxed{x^3y} + \boxed{4xy^2z} + \boxed{C_1(y,z)}$

$x^3 + 8xyz = f_y \Rightarrow \int f_y\,dy = f = \boxed{x^3y} + \boxed{4xy^2z} + \boxed{C_2(x,z)}$

$4xy^2 - e^z = f_z \Rightarrow \int f_z\,dz = f = \boxed{4xy^2z} \boxed{-e^z} + \boxed{C_3(x,y)}$

$$\boxed{f = x^3y + 4xy^2z - e^z + C}$$

Problem 11. (a) Is $\mathbf{F} = \langle x, e^y \sin z, e^y \cos z\rangle$ a conservative vector field? If so, find a potential function for $\mathbf{F}$.

① $\nabla \times \vec{F} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ \partial x & \partial y & \partial z \\ x & e^y \sin z & e^y \cos z \end{vmatrix} = (\cancel{e^y \cos z} - \cancel{e^y \cos z})\,\hat{i} - (\cancel{0} - \cancel{0})\,\hat{j} + (\cancel{0} - \cancel{0})\,\hat{k}$

$= 0 \Rightarrow \vec{F}$ conservative!

② $x = f_x \Rightarrow \int f_x\,dx = \boxed{\frac{x^2}{2}} + \boxed{C_1(y,z)}$

$e^y \sin z = f_y \Rightarrow \int f_y\,dy = \boxed{e^y \sin z} + \boxed{C_2(x,z)}$

$e^y \cos z = f_z \Rightarrow \int f_z\,dz = \boxed{e^y \sin z} + \boxed{C_3(x,y)}$

$$\boxed{f = \frac{x^2}{2} + e^y \sin z + C}$$

(b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2\rangle$, for $1 \le t \le 2$.

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(1))$

$= f(16, 2, 8) - f(1, 1, 2)$

$= \boxed{\left(\frac{16^2}{2} + e^2 \sin(8)\right) - \left(\frac{1}{2} + e \sin(2)\right)}$

$\vec{r}(2) = \langle 2^4, 2, 2 \cdot 2^2\rangle = \langle 16, 2, 8\rangle$

$\vec{r}(1) = \langle 1^4, 1, 2 \cdot 1^2\rangle = \langle 1, 1, 2\rangle$