SECTION 5.3:RATIONAL FUNCTIONS

- Domain - denominator can not be zero
- Rationalizing
- Difference Quotient

$\frac{p(x)}{q(x)}$ ← two polynomials

Pr 1. Which of the following are rational functions?

(a) $f(x) = 3x^2 + 5x - 2$ $= \frac{3x^2+5x+2}{1}$ constants are polynomials

$f(x)$ is rational

(b) $g(x) = x^{-1} + 2x$ Attempt 1: $\frac{x^{-1}+2x}{1}$

$= \frac{1}{x} + \frac{2x}{1} = \frac{1}{x}\frac{1}{1} + \frac{2x}{1}\frac{x}{x} = \frac{1+2x\cdot x}{x}$

$g(x)$ is rational. $= \frac{1+2x^2}{x}$ ✓

(c) $h(x) = \frac{3x-1}{2x+5}$ Yes $h(x)$ is rational

(d) $j(x) = 2^x$ $= \frac{2^x}{1}$ → 2^x is not a polynomial

Not a rational function

$j(x)$ is an exponential function

(e) $h(x) = x^{3.2}$ → not rational

$x^{\frac{32}{10}}$ power is not an integer

power function

cancel out common factors of numerator + denominator

Pr 2. Compute each of the following, and simplify completely:

(a) $\left(\frac{(x+2)^2}{x^2-9}\right)\left(\frac{7(x+9)}{(x-9)}\right) = \frac{7(x+2)^2(x+9)}{(x^2-9)(x-9)}$

$$= \boxed{\frac{7(x+2)^2(x+9)}{(x-3)(x+3)(x-9)}}$$

$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

(b) $\frac{(4x^3-12x^2+9x)}{(x^2-49)} \div \frac{(10x^2-15x)}{(x^2+4x-21)} = \frac{(4x^3-12x^2+9x)(x^2+4x-21)}{(x^2-49)(10x^2-15x)}$

$x^2 = x \cdot x$

$x^3 = x^2 \cdot x$

$\frac{ab}{ad} = \frac{b}{d}$

$$= \frac{\cancel{x}(4x^2-12x+9)\cancel{(x+7)}(x-3)}{(x-7)\cancel{(x+7)}5\cancel{x}(2x-3)}$$

-21: $a = 7$, $b = -3$, $a+b=4$

$\frac{a+b}{a+d} \neq \frac{b}{d}$

$f(x)$ is divisible by $x-c$ if $f(c)=0$

$4(7)^2 - 12(7)+9 \neq 0$

$\approx 200 \neq 0$

check the roots of $4x^2-12x+9$ with the calc.

$4x^2-12x+9 = (2x-3)(2x-3)$

$$\frac{(2x-3)(x-3)}{5(x-7)}$$

(c) $\frac{2}{(x-2)(x+4)} - \frac{x+5}{(x-2)(x+2)}$

$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$

common denominator $= (x-2)(x+4)(x+2)$

$$\frac{(2)(x+2)}{(x-2)(x+4)(x+2)} - \frac{(x+5)(x+4)}{(x-2)(x+2)(x+4)}$$

$$= \frac{2(x+2) - (x+5)(x+4)}{(x-2)(x+4)(x+2)}$$

$$= \frac{-x^2-7x-22}{(x+2)(x+4)(x-2)} = \boxed{-\frac{x^2+7x+22}{(x+2)(x-2)(x+4)}}$$

numerator $=$ $2x+2$ $-(x^2+4x+5x+20)$ $= 2x-2$ $-x^2-9x-20$ $= -x^2-7x-22$

$x = \frac{-7 \pm \sqrt{7^2-4 \cdot 1 \cdot 22}}{2}$

$49-88<0$

no real roots

Denominator $\neq 0$

Pr 3. State the domain of each rational function. Then classify each each domain restriction as the location of a hole or vertical asymptote on the graph of the function. Finally, compute the $x-$ and $y-$intercepts, if possible, of each function.

(a) $f(x) = \dfrac{(3x-2)(2x-5)}{(x-5)(2x+5)}$

Domain: $(x-5)(2x+5) \neq 0$ $\quad ab \neq 0 \to a \neq 0$ and $b \neq 0$

$x-5 \neq 0$ and $2x+5 \neq 0$

$x \neq 5$ and $\frac{2x}{2} \neq \frac{-5}{2} \to x \neq -5/2$

$-\frac{5}{2} < 5$

$$\boxed{\text{Domain: } (-\infty, -5/2) \cup (-5/2, 5) \cup (5, \infty)}$$

hole / asymptote:

$\dfrac{(3x-2)(2x-5)}{(x-5)(2x+5)}$ = doesn't simplify

vertical asymptotes @ $x = -5/2,\ 5$

x-intercept: solve $N(x) = 0$ and $D(x) \neq 0$ $\quad f(x) = \dfrac{N(x)}{D(x)}$

$(3x-2)(2x-5) = 0 \to 3x-2=0$ and $2x-5=0$

$x = 2/3 \qquad x = 5/2$

$\boxed{(2/3, 0) \text{ and } (5/2, 0)}$

y-intercept: set $x=0 \to \dfrac{(3\cdot0-2)(2\cdot0-5)}{(0-5)(2\cdot0+5)} = \dfrac{(-2)(-5)}{(-5)(+5)} = -2/5 \to \boxed{(0, -2/5)}$

(b) $g(x) = \dfrac{(x+3)(x-2)}{(x-2)(x+2)}$ "=" $\dfrac{(x+3)}{(x+2)}$

Domain:
$(x-2)(x+2) \neq 0 \to x-2 \neq 0$ and $x+2 \neq 0$

$\to x \neq 2$ and $x \neq -2$

$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$x = -2$ is a vertical asymptote

plug in $x=2$, $\quad \dfrac{2+3}{2+2} = \dfrac{5}{4}$ $\quad (2, 5/4)$ is a hole

x-intercepts: $(x+3)(x-2) = 0$

$x+3=0 \qquad x-2=0$ ignore this factor

$x = -3$

$\boxed{(-3, 0)}$

y-intercept: "shortcut" plug in $x=0$ into $\dfrac{(x+3)}{(x+2)}$ (unless $D(0) = 0$)

$\dfrac{0+3}{0+2} = 3/2 \to \boxed{(0, 3/2)}$

non-simplified

(c) $h(x) = \frac{-2x}{6x^2 - 8x} = \frac{-2x}{2x(3x-4)}$ " = " $\frac{-1}{3x-4}$ $\qquad \frac{a}{a \cdot b} = \frac{1}{b}$

Domain: $2x(3x-4) \neq 0 \rightarrow 2x \neq 0 \rightarrow x \neq 0$; $3x-4 \neq 0 \rightarrow 3x \neq 4 \rightarrow x \neq 4/3$

$\boxed{\text{Domain: } (-\infty, 0) \cup (0, 4/3) \cup (4/3, \infty)}$

0 not in the domain implies there is no y-intercept ✓

x-intercept: $N(x) = 0$ and $D(x) \neq 0$

$-2x = 0 \rightarrow x = 0$ $\quad D(0) = 0$

hole @ $x=0$

vertical asymptote @ $x = 4/3$

No x-intercept with $x=0$

$\frac{-1}{3 \cdot 0 - 4} = \frac{-1}{-4}$ No x-intercepts.

hole @ $(0, 1/4)$

(d) $j(x) = \frac{3x^2 - 6x + 3}{x^2 - 9}$

$= \frac{3(x^2 - 2x + 1)}{(x-3)(x+3)} = \frac{3(x-1)^2}{(x-3)(x+3)}$ doesn't simplify

$x \neq 3, \quad x \neq -3$

Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Vertical asymptotes @ $x = -3, 3$

x-intercepts: $3(x-1)^2 = 0 \rightarrow (x-1)^2 = 0 \rightarrow x - 1 = 0 \rightarrow x = 1$

$(1, 0)$

y-intercept: $j(0) = \frac{3 \cdot 0^2 - 6 \cdot 0 + 3}{0^2 - 9} = \frac{3}{-9} = -\frac{1}{3}$

$(0, -1/3)$ ✓

Pr 4. Compute and simplify the difference quotient for each function.

$$\to \frac{f(x+h) - f(x)}{h} \leftarrow \text{get rid of } h.$$

(a) $f(x) = -x^2 + 5x - 4$

Step 1: $f(x+h) = -(x+h)^2 + 5(x+h) - 4$

$= -(x^2 + 2xh + h^2) + 5x + 5h - 4$

$= -x^2 - 2xh - h^2 + 5x + 5h - 4$

Step 2: $f(x+h) - f(x) = -x^2 - 2xh - h^2 + 5x + 5h - 4 - (-x^2 + 5x - 4)$

$= -x^2 - 2xh - h^2 + 5x + 5h - 4 + x^2 - 5x + 4$

$= -2xh - h^2 + 5h$

Step 3: $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(-2x - h + 5)}{h} = \boxed{-2x - h + 5}$

(b) $g(x) = \dfrac{3x}{2x-2}$

$\dfrac{a}{b} = \dfrac{1}{b} \cdot a$

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left[\frac{3(x+h)}{2(x+h)-2} - \frac{3x}{2x-2}\right]$$

$$= \frac{1}{h}\left[\frac{3x+3h}{2x+2h-2} - \frac{3x}{2x-2}\right] \leftarrow$$

common den? $(2x-2)(2x+2h-2)$

$$= \frac{1}{h}\left[\frac{(3x+3h)(2x-2)}{(2x+2h-2)(2x-2)} + \frac{-3x(2x+2h-2)}{(2x-2)(2x+2h-2)}\right]$$

$$= \frac{1}{h(2x-2)(2x+2h-2)}\left[(3x+3h)(2x-2) + (-3x)(2x+2h-2)\right]$$

$+(-a) = -a$

$$(6x^2 - 6x + 6xh - 6h - 6x^2 - 6xh + 6x)$$

$$= \frac{-6h}{h(2x-2)(2x+2h-2)} = \frac{-6}{(2x-2)(2x+2h-2)}$$

Section 5.4: Power and Radical Functions

- Power Functions
- Radical Functions
- Domain of Radical Functions based on Index
- Conjugate
- Rationalizing a numerator or denominator

x^{constant}

Pr 1. Which of the following are power or radical functions?

$\sqrt[n]{x^a}$

(a) $f(x) = 3x^2 + 5x - 2$

not a power function

(b) $g(x) = x^{-3}$ → x^{-3} is a power function.

(c) $h(x) = \sqrt{x^7}$ → $\sqrt{x^7}$ is a radical function

(d) $j(x) = 2^x$

x is in the exponent so not a power function

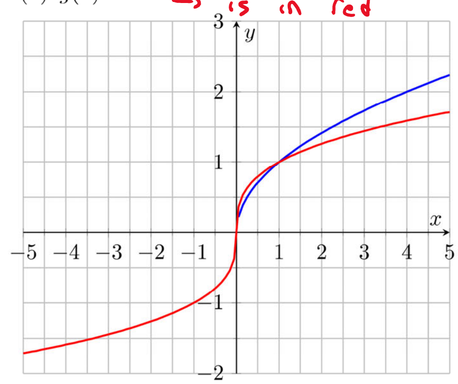
(e) $h(x) = x^{3.2}$ is a power function

(f) Identify the graph of the parent function:

(i) $f(x)\sqrt{x}$ $f(x) = \sqrt{x}$ is in blue $\sqrt{-1}$ DNE

(ii) $g(x) = x^{1/3}$ → is in red

$\sqrt{\text{negatives}}$ DNE



$\sqrt[3]{x}$

even roots of negatives DNE.

odd roots of negatives are "fine"

Pr 2. Rewrite each radical in its equivalent exponent (power) form, assuming x is in the domain of each function.

(a) $\sqrt[5]{-2x^2+4x}$ $=$ $(-2x^2+4x)^{1/5}$ $\qquad \sqrt[n]{a^6} = a^{6/n}$

(b) $6\sqrt{3x^2-8x+2}$ $=$ $6\cdot(3x^2-8x+2)^{1/2}$

2

$\neq \sqrt[6]{3x^2-8x+2}$

(c) $\sqrt[4]{(2-5x)^3}$ $=$ $(2-5x)^{\frac{3}{4}} = (2-5x)^{3/4}$

Pr 3. Rewrite each exponent function in its equivalent radical form, assuming x is in the domain of each function.

(a) $(x^2+3x)^{7/11}$ $=$ $\sqrt[11]{(x^2+3x)^7}$

root 11

(b) $(3x+8)^{9/13}$ $=$ $\sqrt[13]{(3x+8)^9}$

root 13

(c) $2(5x-3)^{7/3}$ $=$ $2\sqrt[3]{(5x-3)^7}$

root 3

Pr 4. State the domain of each function. Write your answer using interval notation. Then determine the x- and y-intercepts, if possible.

(a) $f(x) = \sqrt[6]{3x-28}$

root = 6 is even

$\sqrt[\text{even}]{f(x)} \rightarrow$ require $f(x) \geq 0$

require $3x - 28 \geq 0$

$+28 \quad +28$

$\frac{3x}{3} \geq \frac{28}{3}$

$x \geq \frac{28}{3}$

$\left[\frac{28}{3}, \infty\right)$

y-intercept: $f(0)$

$= \sqrt[6]{3\cdot 0 - 28}$

DNE

No y-intercept

x-intercept: $\sqrt[6]{3x-28} = 0$

$3x - 28 = 0^6 = 0$

$3x = 28$

$x = 28/3$

$\boxed{\left(\frac{28}{3}, 0\right)}$

(b) $g(x) = 2\sqrt[5]{x-5}$ $\rightarrow$ root = 5 is odd

is a poly. $\rightarrow$ Domain: $(-\infty, \infty)$

x-intercept: $2\sqrt[5]{x-5} = 0$

$\sqrt[5]{x-5} = 0 \rightarrow x - 5 = 0$

$x = 5$ ✓

$(5, 0)$

y-intercept: $(0, 2\sqrt[5]{-5})$

set $x=0$

(c) $h(x) = 5(2x-5)^{5/12}$ $= 5\sqrt[12]{(2x-5)^5}$

root is even

need $(2x-5)^5 \geq 0 \rightarrow 2x - 5 \geq 0$

$2x \geq 5$

$x \geq 5/2$

$\left[\frac{5}{2}, \infty\right)$

Y-intercept: set $x=0$ not possible

No y-intercept

x-intercept: $\left(\frac{5}{2}, 0\right)$

Pr 5. State the domain of each function. Write your answer using interval notation.

(a) $f(x) = (3x-4)^{-4/3}$ $= \dfrac{1}{(3x-4)^{4/3}} = \dfrac{1}{\sqrt[3]{(3x-4)^4}}$ $\qquad x^{-a} = \dfrac{1}{x^a}$

$\boxed{(-\infty, 4/3) \cup (4/3, \infty)}$

First "concern" is the radical

radical has odd root 3

→ does'nt effect domain

other concern: Don't divide by 0.

$\sqrt[3]{(3x-4)^4} \neq 0 \rightarrow (3x-4) \neq 0$

$3x \neq 4$

$x \neq 4/3$ ✓

(b) $g(x) = \dfrac{\sqrt{x+2}}{5\sqrt[3]{x+2}}$

Denominator can't be zero → $5\sqrt[3]{x+2} \neq 0$

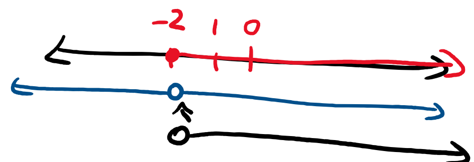
odd root → no issues

even root → $x+2 \geq 0$

$5\sqrt[3]{x+2} \neq 0 \rightarrow \sqrt[3]{x+2} \neq 0 \rightarrow x+2 \neq 0 \rightarrow x \neq -2$

$x+2 \geq 0 \rightarrow x \geq -2$

Domain: $(-2, \infty)$



(c) $h(a) = \dfrac{3a}{\sqrt{a+2}-5}$ → $\sqrt{a+2} - 5 \neq 0$

even root → $a+2 \geq 0$

$a \geq -2$

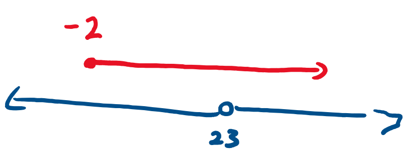
$+\sqrt{a+2} \neq 5$

$a+2 \neq 25$

$a \neq 25-2=23$

$a \neq 23$

$[-2, 23) \cup (23, \infty)$



Pr 6. State the conjugate of each of the following expressions.

(a) $2+\sqrt{x}$ $\rightarrow$ $2-\sqrt{x}$

$a+b\sqrt{d} \xrightarrow{\text{Conjugate}} a-b\sqrt{d}$

$(a+b\sqrt{d})(a-b\sqrt{d})$

$= a^2 - b^2 \cdot d$

(b) $2+3\sqrt{2x-5}$ $\longrightarrow$ $2 - 3\sqrt{2x-5}$

mult. by $\frac{\text{conjugate}}{\text{conjugate}}$

Pr 7. Rationalize each numerator or denominator, as appropriate, and simplify the expression.

(a) $\frac{3x}{\sqrt{x}-3} = \frac{(3x)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} = \frac{3x\sqrt{x}+9x}{(\sqrt{x})^2-3\sqrt{x}+3\sqrt{x}-3\cdot 3}$

$= \boxed{\frac{3x\sqrt{x}+9x}{x^2-9}}$

rationalize the numerator

(b) $\frac{\sqrt{3x-2}+5}{1} \cdot \frac{(\sqrt{3x-2}-5)}{(\sqrt{3x-2}-5)} = \frac{(\sqrt{3x-2})^2-5^2}{\sqrt{3x-2}-5}$

$= \frac{3x-2-25}{\sqrt{3x-2}-5} = \boxed{\frac{3x-27}{\sqrt{3x-2}-5}}$

Pr 8. Compute and simplify the difference quotient for $F(x) = 3\sqrt{2-x}$.

involves square root

key step: multiply by the "conjugate"

$$\frac{F(x+h) - F(x)}{h} = \frac{3\sqrt{2-(x+h)} - 3\sqrt{2-x}}{h}$$

$$= 3\,\frac{\left(\sqrt{2-x-h} - \sqrt{2-x}\right)}{h} \cdot \frac{\left(\sqrt{2-x-h} + \sqrt{2-x}\right)}{\left(\sqrt{2-x-h} + \sqrt{2-x}\right)}$$

$$= \frac{3\left(\sqrt{2-x-h}^{\,2} - \sqrt{2-x}^{\,2}\right)}{h\left(\sqrt{2-x-h} + \sqrt{2-x}\right)} = 3\,\frac{\left(2-x-h-(2-x)\right)}{h\left(\sqrt{2-x-h} + \sqrt{2-x}\right)}$$

$$= \frac{3\left(2-x-h-2+x\right)}{h\left(\sqrt{2-x-h} + \sqrt{2-x}\right)}$$

$$= \frac{-3h}{h\left(\sqrt{2-x-h} + \sqrt{2-x}\right)} = \boxed{\frac{-3}{\sqrt{2-x-h} + \sqrt{2-x}}}$$