Section 3.4: Simplex Method

- Linear programming problem - standard form, introducing slack variables, constructing simplex tableau.
- pivot column, pivot row, and pivot element for a given simplex tableau.
- basic and non-basic variables, optimal solution, leftovers.
- Use technology to perform pivots on a simplex tableau to put the tableau in final form.

Pr 1. Determine if the following linear programming problems are standard maximization problems. If they are, then convert the constraints of the linear programming problem to linear equations with slack variables, and right down the corresponding tableau.

$ax + by + cz \le d$
d is $+$.

(a)

$$\begin{aligned} \text{maximize } & P = 2x + y \\ \text{subject to: } & 2y \le 9 - x \\ & 8 - y \le x \\ & x \ge 0, y \ge 0 \end{aligned}$$

$2y \le 9 - x$
$+x \quad +x$
$x + 2y \le 9$ ✓

$8 - y \le x$
$-8 - x \quad -x - 8$
$-x - y \le -8$
is negative

not a standard maximization problem

(b)

$$\begin{aligned} \text{Maximize } & R = y - x \\ \text{subject to: } & 3y \le 18 - 2x \\ & y - 2x + 10 \ge 0 \\ & x \ge 0, y \ge 0 \end{aligned}$$

$R = y - x$
$R + x - y = 0$
$x - y + R = 0$

$3y \le 18 - 2x$
$+2x \quad +2x$
$2x + 3y \le 18$ ✓

Approach 1: $y - 2x + 10 \ge 0$
$-10 \quad -10$
we could $-1\,(y - 2x \ge -10)$
$-y + 2x \le 10$ ✓
$2x - y \le 10$

This is a standard max. problem

For slack variables, 1 slack variable for each 'non-trivial constraint' not non-negativity constraints

$2x + 3y + s_1 = 18$
$2x - y + s_2 = 10$

$$\begin{array}{cccccc} x & y & s_1 & s_2 & R & \text{con} \end{array}$$
$$\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 0 & 0 & 18 \\ 2 & -1 & 0 & 1 & 0 & 10 \\ \hline 1 & -1 & 0 & 0 & 1 & 0 \end{array}\right]$$

2nd 1st 3rd

Pr 2. For the following simplex tableau, identify the pivot row, pivot column, and pivot element.

(a)

$$\begin{array}{c}\begin{bmatrix} x & y & s_1 & s_2 & P & \vert & \text{constant} \\ 0 & 2 & 1 & 0 & 0 & \vert & 8 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \vert & 5 \\ \hline 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 1 & \vert & 15 \end{bmatrix}\end{array}$$

$8/2 = 4$

$5/(1/2) = 5 \times 2 = 10$

pivot column

most negative entry in the last row

$\boxed{\text{pivot column is 2nd column}}$

constant/pivot column

look for smallest ratio (denominator ≠ 0, or negative)

$\boxed{\text{pivot row is 1st row}}$

pivot entry = 2

(b)

$$\begin{bmatrix} x & y & s_1 & s_2 & s_3 & P & \vert & \text{constant} \\ 1 & 0 & 1 & 0 & 0 & 0 & \vert & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 & \vert & 2 \\ 3 & \frac{1}{3} & 0 & 0 & 1 & 0 & \vert & 200 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & \vert & 0 \end{bmatrix}$$

pivot column = 2nd column

$1/0$ = DNE

$2/-1$ ← not allowed

$200/(1/3) = 200 \times 3 = 600$

pivot row = 3rd row

Pivot entry = $1/3$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

$$a \div \frac{c}{d} = \frac{a}{1} \times \frac{d}{c}$$

(c)

$$\begin{bmatrix} x & y & z & s_1 & s_2 & P & \vert & \text{constant} \\ 0 & 2 & 1 & 0 & 0 & 0 & \vert & 8 \\ 1 & \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & \vert & 5 \\ \hline 0 & \frac{1}{2} & 0 & 2 & \frac{3}{2} & 1 & \vert & 15 \end{bmatrix} \leftarrow$$

No negative entries

There is no pivot column, row or entry.

optimal solution – can't pivot
bottom row all ≥ 0

Pr 3. For the following simplex tableau, identify the basic and non-basic variables. State the solution corresponding to the tableau, and determine if it is an optimal solution.

basic variables: looking for $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

(a)
$$\left[\begin{array}{cccccc|c} x & y & s_1 & s_2 & P & & \text{constant} \\ 0 & 2 & 1 & 0 & 0 & & 8 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & & 5 \\ \hline 0 & \frac{-1}{2} & 0 & \frac{3}{2} & 1 & & 15 \end{array}\right]$$

not optimal

basic variables = x, s_1, P
non-basic var's = every thing else
= y, s_2

$x = 5$
$s_1 = 8$

corresponding solution:
(set non-basic var's $= 0$)
$y = 0$
$s_2 = 0$

$x = 5$
$s_1 = 8$

Solution: $(5, 0)$
$P = 15$
Not optimal!

(b)
$$\left[\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 0 & 0 & 2 \\ 3 & \frac{1}{3} & 0 & 0 & 1 & 0 & 200 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

not optimal

basic var's: s_1, s_2, s_3, P

non-basic var's: x, y

Solution:
$x = y = 0$, solve for others / solve for P
$P = 0$
$(0, 0)$ with $P = 0$.
Not optimal

(left-overs: $s_1 = 1$
$s_2 = 2$
$s_3 = 200$)

additional Tricky example

non-basic
$\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix}$ two 1's

(c)
$$\left[\begin{array}{cccccc|c} x & y & z & s_1 & s_2 & P & \text{constant} \\ 0 & 2 & 1 & 0 & 0 & 0 & 9 \\ 1 & \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & 2 \\ \hline 0 & +\frac{1}{2} & 0 & +2 & +\frac{3}{2} & +1 & 42 \end{array}\right]$$

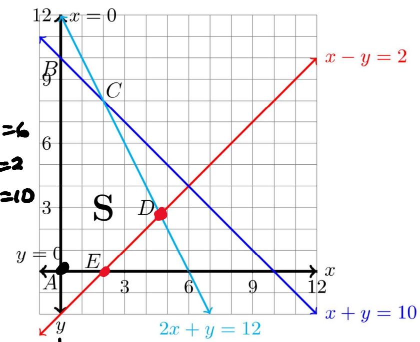
$$\left[\begin{array}{ccc|c} x & z & P & \\ 0 & 1 & 0 & 9 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 42 \end{array}\right]$$

$z = 9$
$x = 2$
$z = 42$

basic var's: x, z, P
non-basic var's: y, s_1, s_2
$y = s_1 = s_2 = 0$
$x = 2$
$z = 9$
$P = 42$

Solution is $(2, 0, 9)$ with
$P = 42$
This is optimal. ✓

Pr 4. Solve using the simplex method. For each tableau, identify the corresponding corner point.



x=y=0

(0,0) point A ←

x	y	s_1	s_2	s_3	P	constant	
2	1	1	0	0	0	12	12/2=6
1	−1	0	1	0	0	2	2/1=2
1	1	0	0	1	0	10	10/1=10
−4	−1	0	0	0	1	0	

2nd row, 1st col,

x	Y	s_1	s_2	s_3	P	const.	
0	3	1	−2	0	0	8	8/3 ✓
1	−1	0	1	0	0	2	DNE ← error "2/−1" "2"
0	2	0	−1	1	0	8	8/2=4
0	−5	0	4	0	1	8	

is this the "final" tableau → No

x=2
y=0
(2,0)
point E

1st row, 2nd column

X	Y	s_1	s_2	s_3	P	con
0	1	1/3	−2/3	0	0	8/3
1	0	1/3	1/3	0	0	14/3
0	0	−2/3	1/3	1	0	8/3
0	0	5/3	2/3	0	1	64/3

← This is the final tableau.

basic vars: x, y, s_3, P

non-basic vars: s_1, s_2

x = 14/3
y = 8/3
P = 64/3

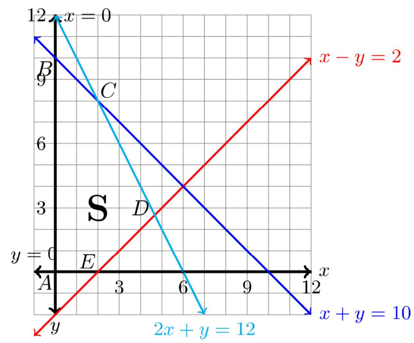
Solution = $\left(\frac{14}{3}, \frac{8}{3}\right)$ with $P = \frac{64}{3}$

point D.

what if
1 1
0 0 } never happens

Solve using the simplex method. For each tableau, identify the corresponding corner point.

$$\left[\begin{array}{cccccc|c} x & y & s_1 & s_2 & s_3 & P & \text{constant} \\ 2 & 1 & 1 & 0 & 0 & 0 & 12 \\ 1 & -1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 1 & 0 & 10 \\ \hline -4 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$



Pr 5. Solve.

Your burger company sells three different types of patty melts - the Big cheesy, the double decker, and the classic. These patty melts all use different amounts of cheese (slices), bread (slices), and patties, as given in the table.

		Cheese	Bread	Patties
x	Big Cheesy	3	2	2
y	Double Decker	2	3	2
z	Classic	1	2	1

The profit for the Big Cheesy is \$1, for the Double Decker is \$2 and for the Classic is \$1. Due to certain agreements, the company can make at most 250 Double Deckers. If the company has 300 slices of cheese, 600 slices of bread, and 800 beef patties, how many of each type of patty melt should be produced in order to maximize the profit? Are there any leftovers?

$x \le y+z$

$x - y - z \le 0$

$2x \le y$

Maximize $P = x + 2y + z$

Subject to:

1) $3x + 2y + z \le 300$

2) $2x + 3y + 2z \le 600$

3) $2x + 2y + z \le 800$

4) $y \le 250$

$x \ge 0,\ y \ge 0,\ z \ge 0$

x = # of Big cheesies

y = # of Double Decker

z = # of classics

P = total profit

$P = x + 2y + z$

$-x - 2y - z + P = 0$

Approach 1: introduce slack variables

Approach 2: compute # of slack variables = # ~~3~~ 4

x	y	z	s_1	s_2	s_3	s_4	P	con	
3	2	1	1	0	0	0	0	300	300/2 = 150
2	3	2	0	1	0	0	0	600	600/3 = 200
2	2	1	0	0	1	0	0	800	800/2 = 400
0	1	0	0	0	0	1	0	250	250/1 = 250
-1	-2	-1	0	0	0	0	1	0	

pivot on 1st row, 2nd column

Final Tableau

x	y	z	s_1	s_2	s_3	s_4	P	con
3/2	1	1/2	1/2	0	0	0	0	150
-5/2	0	1/2	-3/2	1	0	0	0	150
-1	0	0	-1	0	1	0	0	500
-3/2	0	-1/2	-1/2	0	0	1	0	100
2	0	0	1	0	0	0	1	300

$s_1 = 0$ (cheese)

$s_2 = 150$ (bread)

$s_3 = 500$ (patties)

$s_4 = 100$ (other con.)

150 slices of bread + 500 patties left over. no cheese left over

basic var's: y, s_2, s_3, s_4, P

non basic: x, z, s_1

Solution is $(0, 150, 0)$ with $P = 300$

The maximum profit is \$300 from selling 150 Double Deckers.

SECTION 4.1: MATHEMATICAL EXPERIMENTS → set up for rest of chapter 4

- Sample space, **S** - a list of all possible outcomes in the mathematical experiments
- Event - a subset of the sample space
  - Simple Event
  - Certain Event
  - Impossible Event
- Using tree diagrams to determine a sample space in a two-stage experiment
- Venn Diagrams
- Operations on Events
  - Complement, A^C
  - Intersection, $A \cap B$
  - Union, $A \cup B$
- Mutually Exclusive Events

Pr 1. State the sample space for each experiment:

(a) Selecting a letter at random from the word "mathematics" and noting the letter.

$S = \{m, a, t, h, e, i, c, s\}$

(b) Identical ping pong balls are numbered 0 to 10, one ping pong ball is drawn at random, noting the number on the ball.

$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(c) A standard 120-sided die is rolled and it is noted whether the number is a multiple of 3 or is not a multiple of 3.

$S = \{\text{multiple of } 3, \text{ not a multiple of } 3\}$

→ 3, 6, 9, 12 are multiples of 3

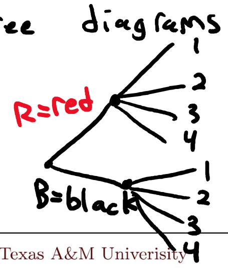
(d) The numbers 0, 1, 2, 3, and 4 are written on separate pieces of paper and put in a hat. Two pieces of paper are drawn at the same time and the product of the numbers is noted.

$S = \{\underset{0\cdot\text{any}}{0}, \underset{1\cdot 2}{2}, \underset{1\cdot 3}{3}, \underset{1\cdot 4}{4}, \underset{2\cdot 3}{6}, \underset{2\cdot 4}{8}, \underset{3\cdot 4}{12}\}$

$1\cdot 1 = 1$ is not in S

(e) A card is drawn from a standard deck of 52-cards, noting the color, and then a fair 4-sided die is rolled, noting which number is on the bottom face.

Tree diagrams



$S = \{(R,1), (R,2), (R,3), (R,4), (B,1), (B,2), (B,3), (B,4)\}$

outcome = (outcome 1st stage, outcome of 2nd stage)

(R,1) is one outcome

an "event" is a subset of S
= a collection of outcomes

Pr 2. Consider the experiment of selecting a letter at random from the word "business" and noting the letter.

(a) State all the simple events for the experiment.

simple event ← has only one outcome

$S = \{b, u, s, i, n, e\}$

simple events: $\{b\}$, $\{u\}$, $\{s\}$, $\{i\}$, $\{n\}$, $\{e\}$

(b) State the certain event for the experiment.

# of simple events = # of outcomes

S itself

$S = \{b, u, s, i, n, e\}$

(c) Give an example of an impossible event for the experiment.

impossible event = $\emptyset = \{\ \}$ = EMPTY in WebAssign

E = "we drew an o". o is not in business

(d) State the total number of possible events.

total number of possible events = $2^{\#\text{ outcomes}}$

$2^6 \neq 2 \cdot 6$ $2^6 = 64$

(e) Write the outcomes in the event, J, "a consonant is draw."

not vowels

$J = \{b, s, n\}$

Pr 3. A card is drawn from a standard deck of 52-cards, noting the color, and then a fair 4-sided die is rolled, noting which number is on the bottom face.

(a) State all the simple events for the experiment.

$\{(R,1)\}$, $\{(R,2)\}$, $\{(R,3)\}$, $\{(R,4)\}$ → 8 outcomes

$\{(B,1)\}$, $\{(B,2)\}$, $\{(B,3)\}$, $\{(B,4)\}$

$\{R\}$ is not a simple event

(b) State the certain event for the experiment.

$$S = \{(R,1), (R,2), (R,3), (R,4), (B,1), (B,2), (B,3), (B,4)\}$$

(c) Give an example of an impossible event for the experiment.

E = "we drew a yellow card".

$E = \{\ \}$

(d) State the total number of possible events.

$$2^{\#\text{ outcomes}} = 2^8 = 256 \text{ possible events}$$

(e) Write the outcomes in the event, M, "a red card is drawn or the die lands on an even number."

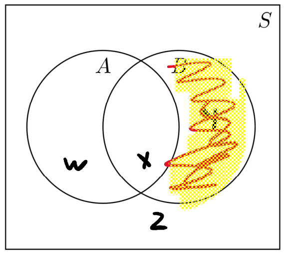
$$M = \{(R,1), (R,2), (R,3), (R,4), (B,2), (B,4)\}$$

Pr 4. Let A and B be two events of the sample space, S. $S = \{w, x, y, z\}$

Use a two-circle Venn diagram to illustrate which region(s) contain the outcomes of the resulting events.

a. $B \cap A^C$

$A = \{w, x\}, \quad B = \{x, y\}$



A^c = complement = outcomes not in A

$A^c = \{y, z\}$

$E \cap F$ = outcomes in both

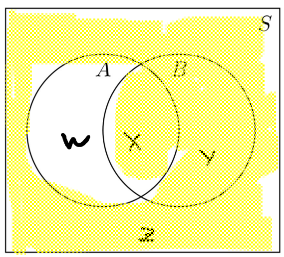
intersection

$B \cap A^c = \{y\}$

$E \cup F = \{x$ in E or x in $F\}$

inclusive (or both)

b. $(A \cap B) \cup A^C$



$(A \cap B) \cup A^c$

union

$A = \{w, x\}, \quad B = \{x, y\}$

$A^c = \{y, z\}$

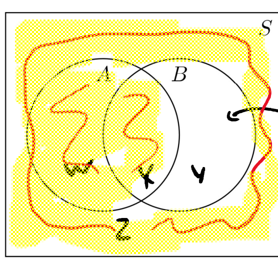
$A \cap B = \{x\}$

$(A \cap B) \cup A^c = \{x\} \cup \{y, z\}$

everything but w $= \{x, y, z\}$

c. $(A^C \cap B)^C$

complement last



not here

$A^c \cap B = \{w, x\}^c \cap \{x, y\}$

$= \{y, z\} \cap \{x, y\}$

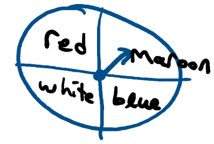
$= \{y\}$

$(A^c \cap B)^c = \{y\}^c = \{w, x, z\}$

$S = \{ (1,R), (2,R), \ldots$

Pr 5. An experiment consists of rolling a six-sided die, noting the number showing uppermost and then spinning a spinner with four equal regions (red, white, blue, and maroon), noting the color.

R = red
H = white
M = maroon
B = blue



Let

V := the event "a number greater than 3 is rolled"

W := the event "an odd is rolled"

X := the event "the spinner lands on blue"

Y := the event "the spinner lands on a color other than maroon" (red, white, blue)

Z := the event "the spinner lands on white or maroon."

$\cup$

(a) Write the symbolic notation for the event, D, that "an even is rolled or the spinner lands on white or maroon."

W^c

$\cup$

$$D = (W^c) \cup Z$$

$$D = \{ (2,R), \ldots \}$$

(b) Write the symbolic notation for the event, H, that "a number less than or equal to 3 is rolled or the spinner lands on a color other than maroon but not blue."

V^c

Y "and" X^c

$$V^c \cup (Y \cap X^c)$$

(c) Describe the event $Y^C \cap W$.

a

d

Y^c = spinner lands on maroon

W = an odd is rolled

The spinner lands on maroon and an odd is rolled.

(d) Describe the event $W \cup (Y^C \cup Z^C)$

not white and not maroon

not (A ~~and~~ or B)

" and odd is rolled or landed on the spinner blue, red, or maroon." = not A and not B

(e) Are event V and event W mutually exclusive? Explain why or why not. $V \cap W = \emptyset$

$V \cap W$ = an odd greater than 3 is rolled

$V \cap W = \{ (5,R), (5,B), (5,H), (5,M) \}$

$V \cap W \neq \emptyset$,

So they are not mutually exclusive.