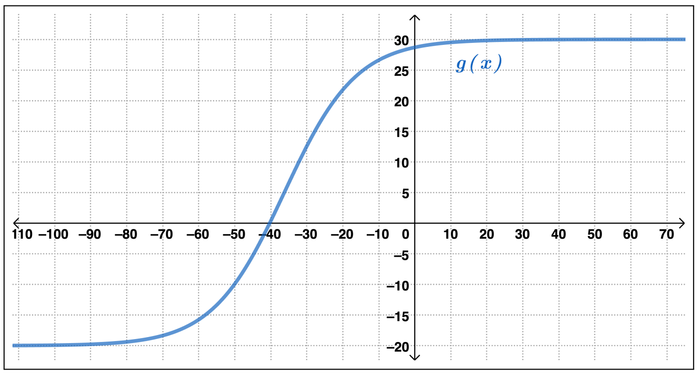
SESSION 2: SECTIONS 1-3 AND 1-4

1. Use the graphs below to estimate the given limits.

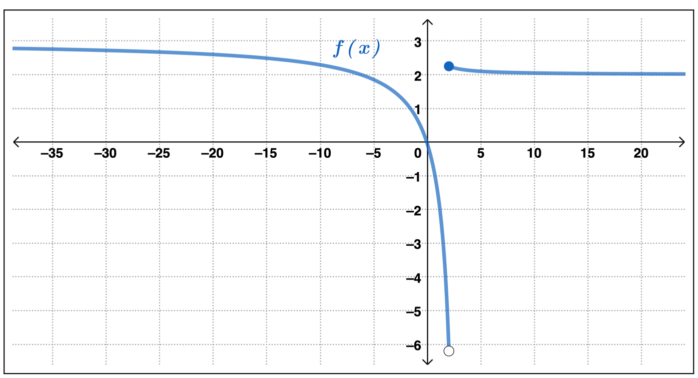
   (a) Estimate $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.



$\lim_{x\to\infty} g(x) \approx 30$

$\lim_{x\to-\infty} g(x) \approx -20$

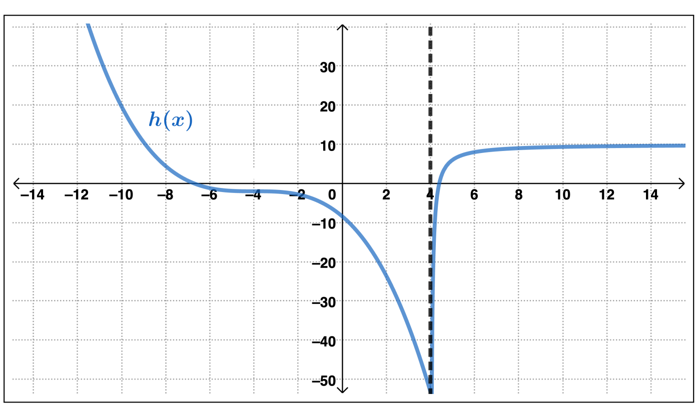
   (b) Estimate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.



$\lim_{x\to\infty} f(x) \approx 2$

$\lim_{x\to-\infty} f(x) \approx 3$

   (c) Estimate $\lim_{x\to\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.



$\lim_{x\to\infty} h(x) \approx 10$

$\lim_{x\to-\infty} h(x)$ DNE, but we can say $\lim_{x\to-\infty} h(x) \to \infty$.

Finding Limits at Infinity Algebraically

- **Rational Functions:** To find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, for rational functions, we divide both the numerator and denominator by the term in the denominator with the highest power of x. Then simplify the function and use the fact that if n is a positive integer, then $\lim_{x\to\infty} \frac{1}{x^n} = 0$ and $\lim_{x\to-\infty} \frac{1}{x^n} = 0$.

- **Fractions containing Exponential Functions:** Use the fact that if n is a positive integer, then
  (a) $\lim_{x\to\infty} e^{nx} \to \infty$ and $\lim_{x\to-\infty} e^{nx} = 0$.
  (b) $\lim_{x\to\infty} e^{-nx} = 0$ and $\lim_{x\to-\infty} e^{-nx} \to \infty$.

2. Use algebric methods to find $\lim_{x\to\infty} \frac{4x^3 - 8x^{10} + 4x^6}{8x^2 - 7x + 5}$ divide by x^2

$$\lim_{x\to\infty} \frac{\frac{4x^3}{x^2} - \frac{8x^{10}}{x^2} + \frac{4x^6}{x^2}}{\frac{8x^2}{x^2} - \frac{7x}{x^2} + \frac{5}{x^2}} = \lim_{x\to\infty} \frac{4x - 8x^8 + 4x^4}{8 - \frac{7}{x} + \frac{5}{x^2}}$$

$$= \frac{\overbrace{\lim_{x\to\infty} 4x}^{\infty} - \overbrace{\lim_{x\to\infty} 8x^8}^{\infty} + \overbrace{\lim_{x\to\infty} 4x^4}^{\infty}}{\underbrace{\lim_{x\to\infty} 8}_{8} - \underbrace{\lim_{x\to\infty} \frac{7}{x}}_{0} + \underbrace{\lim_{x\to\infty} \frac{5}{x^2}}_{0}}$$

} very large negative

} 8

So $\boxed{\lim_{x\to\infty} \frac{4x^3 - 8x^{10} + 4x^6}{8x^2 - 7x - 5} \to -\infty}$

3. Use algebraic methods to find $\lim_{x\to-\infty} \frac{x^2 - x - 72}{18 + 4x^2 - 38x}$ divide by x^2

$$\lim_{x\to-\infty} \frac{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{72}{x^2}}{\frac{18}{x^2} + \frac{4x^2}{x^2} - \frac{38x}{x^2}} = \lim_{x\to-\infty} \frac{\overbrace{1}^{1} - \overbrace{\frac{1}{x}}^{0} - \overbrace{\frac{72}{x^2}}^{0}}{\underbrace{\frac{18}{x^2}}_{0} + \underbrace{4}_{4} - \underbrace{\frac{38}{x}}_{0}} = \boxed{\frac{1}{4}}$$

4. Use algebric methods to find $\lim_{x\to\infty} \frac{5e^{7x} - 4e^{-2x}}{e^{-2x} + 9e^{8x}}$

$e^{8x} \to \infty$ as $x \to \infty$ so divide by e^{8x}

$$\lim_{x\to\infty} \frac{\frac{5e^{7x}}{e^{8x}} - \frac{4e^{-2x}}{e^{8x}}}{\frac{e^{-2x}}{e^{8x}} + \frac{9e^{8x}}{e^{8x}}} = \lim_{x\to\infty} \frac{\frac{5}{e^{x}} \to 0 - \frac{4}{e^{10x}} \to 0}{\frac{1}{e^{10x}} \to 0 + 9 \to 9} = \frac{0}{9} = \boxed{0}$$

Method for Determining Holes and Vertical Asymptotes of Rational Functions

(a) Factor the numerator and denominator. Divide any common factors.

(b) The factors in the denominator that *divide completely* will determine the holes. Set each factor in the denominator that divides completely equal to zero to find the x-value of the **hole**.

(c) The factors that *remain in the denominator* (and denominator only!) will determine the vertical asymptotes. Again, set each factor that remains in the denominator equal to zero to find the x-value of the **vertical asymptote**.

5. Find any (a) horizontal asymptotes, (b) holes, and (c) vertical asymptotes of the functions given below. If there are no horizontal asymptotes, describe he end behavior using limit notation. For each vertical asymptote, describe the infinite behavior using limit notation.

(a) $f(x) = \frac{(x+8)(x-9)}{(x-9)(4x-2)} = \frac{x^2 - x - 72}{4x^2 - 38x + 18}$

Horizontal Asy.

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{72}{x^2}}{\frac{4x^2}{x^2} - \frac{38x}{x^2} + \frac{18}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1 - \frac{1}{x} \to 0 - \frac{72}{x^2} \to 0}{4 - \frac{38}{x} \to 0 + \frac{18}{x^2} \to 0} = \frac{1}{4}$$

$$\boxed{y = \frac{1}{4}}$$

$(x-9)$ divides out completely from denominator. So hole at $x = 9$. $\frac{9+8}{4(9)-2} = \frac{17}{36-2} = \frac{17}{34} = \frac{1}{2}$

$\boxed{\text{hole at } (9, \frac{1}{2})}$

$(4x-2)$ remains in denominator, so

Vertical asymptote where $4x-2=0$

$4x = 2$

$x = \frac{1}{2}$

$\boxed{\text{Vertical asymptote } x = \frac{1}{2}}$

(b) $f(x) = \dfrac{(2x+7)^2(x-4)}{(x+3)(2x+7)} = \dfrac{4x^3+12x^2-63x-196}{2x^2+13x+21}$

divide by x^2

horizontal asy

$$\lim_{x\to\infty} \frac{\frac{4x^3}{x^2}+\frac{12x^2}{x^2}-\frac{63x}{x^2}-\frac{196}{x^2}}{\frac{2x^2}{x^2}+\frac{13x}{x^2}+\frac{21}{x^2}} = \lim_{x\to\infty} \frac{4x+12-\frac{63}{x}-\frac{196}{x^2}}{2+\frac{13}{x}+\frac{21}{x^2}} \to \infty$$

$\boxed{\text{No horizontal asymptotes; as } x\to\pm\infty, \text{ the function values } \to\infty.}$

Holes: $(2x+7)$ divides out completely from denominator. $\boxed{(-7/2, 0)}$

$2x+7=0$

$2x=-7$

$x=-\frac{7}{2}$

substitute $x=-\frac{7}{2}$ into

$$\frac{(2x+7)(x-4)}{(x+3)} = \frac{(2(-7/2)+7)(-7/2-4)}{(-7/2+3)} = \frac{0}{-.5} = 0$$

vertical asymptotes: $(x+3)$ remains in denominator after simplification.

So there is V.A. @ $\boxed{x=-3}$

(c) $f(x) = \dfrac{(5x-17)\cancel{(x+4)}}{(3x-8)(x+4)^{\cancel{2}}} = \dfrac{5x^2+3x-68}{3x^3+16x^2-16x-128}$

divide by x^3

horizontal asymptotes

$$\lim_{x\to\infty} \frac{\frac{5x^2}{x^3}+\frac{3x}{x^3}-\frac{68}{x^3}}{\frac{3x^3}{x^3}+\frac{16x^2}{x^3}-\frac{16x}{x^3}-\frac{128}{x^3}} = \lim_{x\to\infty} \frac{\frac{5}{x}+\frac{3}{x^2}-\frac{68}{x^3}}{3+\frac{16}{x}-\frac{16}{x^2}-\frac{128}{x^3}} = \frac{0}{3} = 0$$

$\boxed{y=0}$

holes: $\boxed{\text{none}}$

vertical asymptotes:

$3x-8=0$

$3x=8$

$\boxed{x=8/3}$

$x+4=0$

$\boxed{x=-4}$

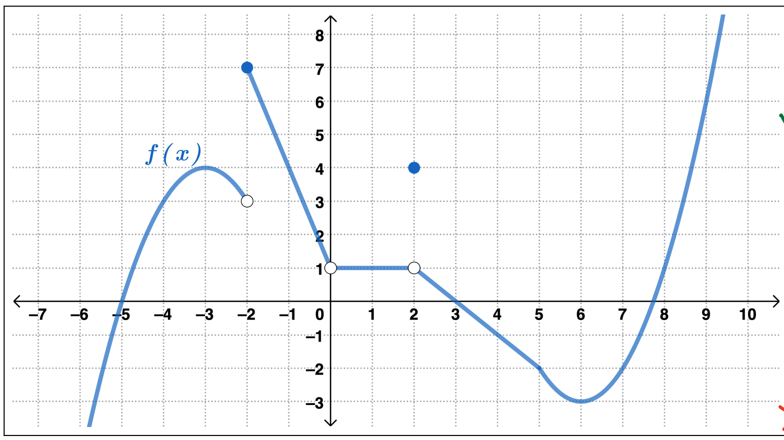
A function f is **continuous at a point** where $x = c$ if and only if the following three conditions are satisfied:

I. $f(c)$ is defined.

II. $\lim_{x \to c} f(x) =$ exists.

III. $\lim_{x \to c} f(x) = f(c)$.

6. Use the graph of f to determine the x-value(s) where f is discontinuous. State the condition in the definition of continuity at a point that fails first at each x-value.



$x = 2$

✓ I. $f(2) = 4$

✓ II. $\lim_{x \to 2^-} f(x) = 1$, $\lim_{x \to 2^+} f(x) = 1$ } $1 = 1$

$\lim_{x \to 2} f(x) = 1$

✗ III. $1 \neq 4$, so $\lim_{x \to 2} f(x) \neq f(2)$

$\boxed{\text{f(x) is discontinuous at } x = 2 \text{ and fails criteria III first.}}$

$x = -2$

✓ I. $f(-2) = -2$

✗ II. $\lim_{x \to -2^-} f(x) = -2$, $\lim_{x \to -2^+} f(x) = 7$ } $-2 \neq 7$

So $\lim_{x \to -2} f(x)$ DNE.

$\boxed{\text{f(x) is discontinuous at } x = -2 \text{ and fails criteria II first.}}$

$x = 0$

✗ I. $f(0)$ is not defined

$\boxed{\text{f(x) is discontinuous at } x = 0 \text{ and fails criteria I first.}}$

7. Determine if the functions below are continuous at the given value of c. If the function is not continuous at $x = c$, also state the condition in the definition of continuity at a point that fails first mathematically.

(a) $f(x) = \frac{2x^2 - x - 15}{x^2 - x - 6} = \frac{(2x+5)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)}$, $c = 10$

note: hole @ $x=3$

Vertical asy @ $x=-2$

These are possible x-values where $f(x)$ could be discontinuous.

$f(x) \approx \frac{2x+5}{x+2}$ with hole at $\left(3, \frac{11}{5}\right)$

✓ Criteria I: $f(10) = \frac{2(10)+5}{10+2} = \frac{25}{12}$

✓ Criteria II: $\lim_{x \to 10} f(x) = \frac{2(10)+5}{10+2} = \frac{25}{12}$

✓ Criteria III: $\lim_{x \to 10} f(x) = f(10) \Rightarrow \frac{25}{12} = \frac{25}{12}$

$\boxed{\text{So } f(x) \text{ is continuous at } x=10}$

(b) $f(x) = \begin{cases} 4x+7 & x < 2 \\ 0 & x = 2 \\ 9x-3 & x > 2 \end{cases}$, $c = 2$

✓ Criteria I: $f(2) = 0$

✓ Criteria II: $\lim_{x \to 2^-} 4x+7 = 4(2)+7 = 8+7 = 15$

$\lim_{x \to 2^+} 9x-3 = 9(2)-3 = 18-3 = 15$

So $\lim_{x \to 2} f(x) = 15$

✗ Criteria III: $f(2) \neq \lim_{x \to 2} f(x)$ because $0 \neq 15$.

$\boxed{f(x) \text{ is not continuous at } x=2\text{, it fails at } 3^{rd} \text{ criteria.}}$

Polynomials, rational functions, power functions, exponential functions, logarithmic functions, and combinations of these are continuous on their domain.

Domain Restrictions

(a) The denominator must be nonzero.

(b) The argument of an even root must be nonnegative.

(c) The argument of a logarithm (of any base) must be positive.

8. Using algebraic methods to determine the intervals on which each functions is continuous. Write your answer using interval notation.

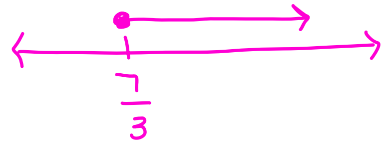
(a) $f(x) = \sqrt{3x-7}$

$\boxed{[7/3, \infty)}$

$3x - 7 \geq 0$

$3x \geq 7$

$x \geq 7/3$



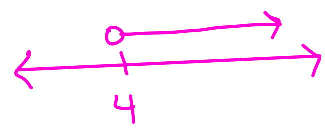
(b) $f(x) = \ln(2x-8)$

$\boxed{(4, \infty)}$

$2x - 8 > 0$

$2x > 8$

$x > 8/2 = 4$



(c) $f(x) = \dfrac{x^2-3x-10}{\sqrt[6]{-2x-7}}$

$\boxed{(-\infty, -\frac{7}{2})}$

$-2x - 7 > 0$

$-2x > 7$

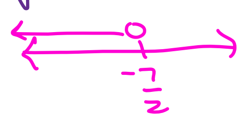
$x < -\frac{7}{2}$

$x^2 - 3x - 10$ is continuous everywhere

$\sqrt[6]{-2x-7} \Rightarrow$ in a denom., so $\neq 0$

even root radical ≥ 0

> 0



(d) $f(x) = \dfrac{4\ln(x-8)}{\sqrt{5x-3}}$

$(8, \infty)$

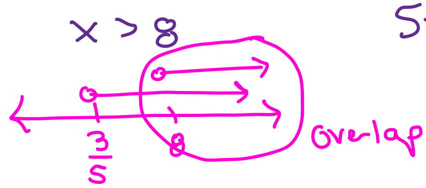
$x - 8 > 0$

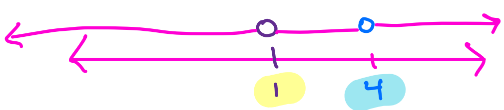
$x > 8$

$5x - 3 > 0 \Longrightarrow$ even root and in denomin.

$5x > 3$

$x > \frac{3}{5}$





$f(x)$ is continuous on $(-\infty, 1) \cup (1, 4) \cup (4, \infty)$

(e) $f(x) = \begin{cases} \frac{4}{x-1} & x \le 2 \\ x^2 & 2 < x < 4 \\ -8\log_2(x+12) & x \ge 4 \end{cases}$

not continu. at $x=1$

① check each function

$\frac{4}{x-1}$ is discontinuous at $x = 1$

✓ x^2, no points of discontinuity

✓ $8\log_2(x+12) \Rightarrow$ continuous on $(-12, \infty)$ + only defined on $x \ge 4$

$x+12 > 0$

$x > -12$

② check cutoff point $x = 2, 4$

$x=2$ ✓ I. $f(2) = \frac{4}{2-1} = \frac{4}{1} = 4$

✓ II. $\lim_{x\to 2^-} f(x) = \frac{4}{2-1} = 4$

$\lim_{x\to 2^+} f(x) = 2^2 = 4$

$\lim_{x\to 2} f(x) = 4$

✓ III. $f(2) = \lim_{x\to 2} f(x)$, $4 = 4$

$\boxed{\text{continuous at } x=2}$

$x = 4$ ✓ I. $f(4) = -8\log_2(4+12)$

$= -8\log_2(16)$

$= -8(4)$

$= -32$

$\boxed{f(x) \text{ is discontinuous at } x=4}$

✗ II. $\lim_{x\to 4^-} x^2 = 4^2 = 16$

$\lim_{x\to 4^+} -8\log_2(x+12) = -8\log_2(4+12) = -8\log_2 16$

$= -8(4)$

$= -32$

$\lim_{x\to 4} f(x)$ DNE: fails criteria II.

9. Given $f(x) = \begin{cases} 2x^2 + k & x < 2 \\ 3x - 8 & x \ge 2 \end{cases}$, answer the questions that follow.

(a) Determine if $f(x)$ will be continuous on $(-\infty, \infty)$ if $k = 4$.

$f(x) = \begin{cases} 2x^2 + 4 & x < 2 \\ 3x - 8 & x \ge 2 \end{cases}$

✓ I. $f(2) = 3(2) - 8 = 6 - 8 = -2$

✗ II. $\lim_{x\to 2^-} 2x^2 + 4 = 2(2)^2 + 4 = 8 + 4 = 12$

$\lim_{x\to 2^+} 3x - 8 = 3(2) - 8 = 6 - 8 = -2$

$\lim_{x\to 2} f(x)$ DNE because $12 \ne -2$.

$\boxed{\text{Not continuous on } (-\infty,\infty) \text{ if } K=4}$

(b) Determine if $f(x)$ will be continuous on $(-\infty, \infty)$ if $k = -1$.

$f(x) = \begin{cases} 2x^2 - 1 & x < 2 \\ 3x - 8 & x \ge 2 \end{cases}$

✓ I. $f(2) = 3(2) - 8 = 6 - 8 = -2$

✗ II $\lim_{x\to 2^-} (2x^2 - 1) = 2(2)^2 - 1 = 8 - 1 = 7$

$\lim_{x\to 2^+} 3x - 8 = -2$

$\lim_{x\to 2} f(x)$ DNE

$\boxed{\text{Not continuous on } (-\infty,\infty) \text{ if } K=-1}$

(c) Find the value(s) of k that make(s) the function continuous for all real numbers. If there is no such value of k, explain why.

We know $f(2) = -2$.

We need $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = -2$

We need $f(2) = \lim_{x\to 2} f(x)$

So we need

$\lim_{x\to 2^-} f(x) = -2$

$\lim_{x\to 2^-} (2x^2 + K) = -2$

$2(2)^2 + K = -2$

$8 + K = -2$

$\boxed{K = -10}$