7.4: BASIC THEORY OF SYSTEMS OF 1ST-ORDER LINEAR EQUATIONS

Review

- **Existence and uniqueness**: Consider the initial value problem

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

  If all the entries of $P(t)$ and $g(t)$ are continuous functions on an open interval $I = (a, b)$, then there exists a unique solution to the initial value problem on the interval I.

- **Principle of superposition**: If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions to the differential equation $\mathbf{x}' = P(t)\mathbf{x}$, then

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

  is also a solution.

- **Wronskian for vector functions**: If $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ are all n-vectors, then their Wronskian is defined as

$$W[\mathbf{x}^{(1)}, ..., \mathbf{x}^{(n)}](t) = \det \mathbf{X}(t),$$

  where $\mathbf{X}(t)$ is the matrix whose columns are $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$.

- **Fundamental set of solutions:** Suppose $P(t)$ is an $n \times n$ matrix. Then, $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ is a fundamental set of solutions if their Wronskian is nonzero.

- **General solution**

- **Abel's theorem**: If $\mathbf{x}^{(1)}$, ..., $\mathbf{x}^{(n)}$ are solutions to $\mathbf{x}' = P(t)\mathbf{x}$ on an interval I, then their Wronskian is either always zero or never zero on I.

  Practical consequence: You only need to check the Wronskian at a single point in the interval where the solution exists.

Exercise 1

Where is the following initial value problem guaranteed to have a unique solution?

$$\mathbf{x}' = \begin{bmatrix} 3 & -t^2+2 \\ \ln(t) & \cos(t) \end{bmatrix} \mathbf{x} + \begin{bmatrix} (t-4)^{-3} \\ 14e^t \end{bmatrix}, \qquad \mathbf{x}(1) = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

Exercise 2

Where is the following initial value problem guaranteed to have a unique solution?

$$\mathbf{x}' = \begin{bmatrix} \tan(t) & \pi \\ \frac{3}{t} & 7t \end{bmatrix} \mathbf{x} + \begin{bmatrix} 8 \\ \sqrt{t+9} \end{bmatrix}, \qquad \mathbf{x}(-\pi/3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Exercise 3

Consider the system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix} \mathbf{x}.$$

Is the following a fundamental set of solutions?

$$\left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t}, \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{t} \right\}$$

Exercise 4

Consider the system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 1/2 & 0 \\ 1 & -1/2 \end{bmatrix} \mathbf{x}.$$

Is the following the general solution?

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t/2} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t/2}$$

7.5: HOMOGENEOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

Review

- How to solve a homogeneous linear system with constant coefficients (when you have distinct real eigenvalues)
  1. Assume your solution has the form $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$.
  2. Plug this in to get an eigenvalue problem.
  3. Solve for the eigenvalues r_1 and r_2 and the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
  4. The general solution is $c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$.
- **Phase plane/portrait**: A phase plane/portrait is essentially a 2D version of the phase line. It shows you where the solution moves as time passes.
- An **equilibrium point** is a point where if you start there, you will remain there forever. The origin is always an equilibrium point of the differential equation system $\mathbf{x}' = A\mathbf{x}$.
- **Stability** of equilibrium points
  - **Asymptotically stable**: If you start near the equilibrium point, you will be sucked into it as $t \to \infty$.
  - **Stable**: If you start near the equilibrium point, you will stay near it.
  - **Unstable**: There is at least one point near the equilibrium point that goes away from the equilibrium point.

Exercise 5

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \mathbf{x}$$

Solve the initial value problem when $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Draw this solution on the phase plane and sketch the graph of $x_1(t)$ and $x_2(t)$.

Exercise 6

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x_1' = -5x_1 + 4x_2$$
$$x_2' = \frac{3}{2}x_1 - 4x_2$$

Exercise 7

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\begin{aligned} x' &= 2x + 2y \\ y' &= x + 3y \end{aligned}$$

7.6: COMPLEX EIGENVALUES

Review

- To solve the system $\mathbf{x}' = A\mathbf{x}$ when you have complex eigenvectors:
  - Solve for just **one** of the eigenvectors.
  - Separate $\boldsymbol{\xi} e^{rt}$ into its real and imaginary parts.
  - The real and imaginary parts form a fundamental set of solutions.
    - (Assuming that A is 2×2. If A is larger, than there are also more solutions.)

Exercise 8

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x' = 3x + y$$
$$y' = -2x + y$$

Exercise 9

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

Exercise 10

Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin. Solve the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$.

$$\mathbf{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \mathbf{x}$$