



TEXAS A&M UNIVERSITY
Math Learning Center

Math 251 - Fall 2023
“HANDS ON GRADES UP”
T, DEC 5, 7-9 ILCB 229
FINAL EXAM REVIEW
COVERING CHAPTER 16 AND
SECTION 14.7

PLEASE SCAN THE QR CODE BELOW



We will begin at 7PM. A problem will be displayed on the table monitors. Collaborate with your table on how to solve each problem. If you have a question, raise your hand. At the end of a predetermined number of minutes, the solutions will be displayed on the wall monitors. Feel free to take a picture of the solution, as the solutions are not posted.

This non solved problem set is posted at <https://mlc.tamu.edu/by-course/math-251Hands-on,-Grades-up>

Problem 1. Evaluate $\int_C (x^2 + y^2 + z^2) ds$, where C is parameterized by $\mathbf{r}(t) = \langle t, \cos 2t, \sin 2t \rangle$, $0 \leq t \leq 2\pi$.

notice the differential, $ds = |\mathbf{r}'(t)| dt$.

General formula: $\int_C f(x, y, z) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt$,

where $C = \mathbf{r}(t)$, $a \leq t \leq b$

$$\mathbf{r}'(t) = \langle 1, -2\sin 2t, 2\cos 2t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{1 + 4\sin^2 2t + 4\cos^2 2t} = \sqrt{5}$$

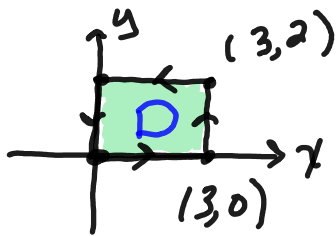
$$\int_C (x^2 + y^2 + z^2) ds = \int_0^{2\pi} (t^2 + \cos^2 2t + \sin^2 2t) \sqrt{5} dt$$

$$= \sqrt{5} \int_0^{2\pi} (t^2 + 1) dt$$

$$= \sqrt{5} \left(\frac{t^3}{3} + t \right) \Big|_0^{2\pi}$$

$$= \sqrt{5} \left(\frac{8\pi^3}{3} + 2\pi \right)$$

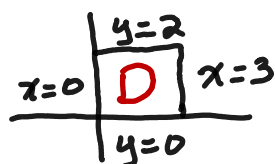
Problem 2. Evaluate $\int_C y^2 dx + xy dy$, where C is the positively oriented rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 2)$, and $(0, 2)$.



- we do not want to solve this as a line integral since there are 4 lines.
- Notice C is a closed positively oriented curve, so we can use Green's Theorem.

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA, \text{ where } D \text{ is the region bounded by } C.$$

$$\int_C \underbrace{y^2}_{P} dx + \underbrace{xy}_{Q} dy = \iint_D (y - 2y) dA = \iint_D (-y) dA$$



$$D: 0 \leq x \leq 3 \\ 0 \leq y \leq 2$$

$$= - \int_0^2 \int_0^3 y dx dy$$

$$= - \int_0^2 y x \Big|_{x=0}^{x=3} dy$$

$$= - \int_0^2 3y dy$$

$$= - \frac{3y^2}{2} \Big|_0^2 = \boxed{-6}$$

Problem 3. Given $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ and $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $0 \leq t \leq \frac{\pi}{2}$.

Hint: Find $\text{curl}(\mathbf{F})$ to see if $\vec{F}$ is conservative.

$$\text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xe^z & \cos y & 2x^2e^z \end{vmatrix} = \langle 0, -(4xe^z - 4xe^z), 0 \rangle = \vec{0}$$

fundamental theorem
of line integrals

$\vec{F}$ is conservative, so we can use FTLI

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\vec{r}(\frac{\pi}{2})) - f(\vec{r}(0)) = f(1, \frac{\pi}{2}, 0) - f(0, 0, 1),$$

where $f(x, y, z)$ is the potential function.

$$\vec{F} = \nabla f \rightarrow \langle 4xe^z, \cos y, 2x^2e^z \rangle = \langle f_x, f_y, f_z \rangle$$

$$f(x, y, z) = \int 4xe^z dx = 2x^2e^z + g(y, z)$$

$$f(x, y, z) = \int \cos y dy = \sin y + h(x, z)$$

$$f(x, y, z) = \int 2x^2e^z dz = 2x^2e^z + k(x, y)$$

$$f(x, y, z) = 2x^2e^z + \sin y$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, \frac{\pi}{2}, 0) - f(0, 0, 1)$$

$$= 2 + 1 - (0) = \boxed{3}$$

Problem 4. Evaluate $\int_C (x - z + y) ds$, where C is the line segment from $(2, 1, 1)$ to $(3, -1, 0)$.

Parameterize the line segment:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$$= \langle 2, 1, 1 \rangle + t \langle 1, -2, -1 \rangle, \quad 0 \leq t \leq 1$$

why? if $t=0$, do you get the initial point? If $t=1$, do you get the terminal point?

$$\vec{r}(t) = \langle 2+t, 1-2t, 1-t \rangle, \quad 0 \leq t \leq 1$$

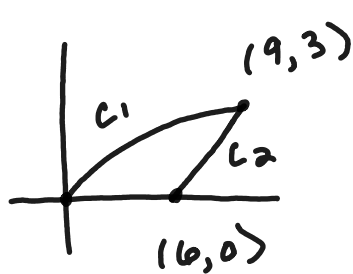
$$\vec{r}'(t) = \langle 1, -2, -1 \rangle, \quad \text{and } |\vec{r}'(t)| = \sqrt{6}$$

$$\int_C (x - z + y) ds = \int_0^1 (2+t - (1-t) + 1-2t) \sqrt{6} dt$$

$$= \sqrt{6} \int_0^1 2 dt$$

$$= \boxed{2\sqrt{6}}$$

Problem 5. Set up but do not evaluate $\int_C xy \, dx + 2y \, dy$, where C is the arc of the curve $y = \sqrt{x}$ from $(0,0)$ to the point $(9,3)$, then the line segment from the point $(9,3)$ to the point $(6,0)$.



$$C_1: y = \sqrt{x} \quad \text{Let } x = t, y = \sqrt{t}, \quad 0 \leq t \leq 9$$

$$dx = dt, \quad dy = \frac{1}{2\sqrt{t}} dt$$

C_2 : Line from $(9,3)$ to $(6,0)$

$$r(t) = \vec{r}_0 + t\vec{v} = \langle 9, 3 \rangle + t\langle -3, -3 \rangle$$

$$x = 9 - 3t, \quad y = 3 - 3t, \quad 0 \leq t \leq 1$$

$$dx = -3 \, dt, \quad dy = -3 \, dt$$

$$\int_C xy \, dx + 2y \, dy = \int_{C_1} xy \, dx + 2y \, dy + \int_{C_2} xy \, dx + 2y \, dy$$

$$= \int_0^9 \underbrace{(t)}_x \underbrace{(\sqrt{t})}_y \underbrace{(dt)}_{dx} + 2 \underbrace{\sqrt{t}}_y \underbrace{\left(\frac{1}{2\sqrt{t}}\right) dt}_{dy}$$

+

$$\int_0^1 \underbrace{(9-3t)}_x \underbrace{(3-3t)}_y \underbrace{(-3 \, dt)}_{dx} + 2 \underbrace{(3-3t)}_y \underbrace{(-3 \, dt)}_{dy}$$

Problem 6. Find the work done by the force field $\mathbf{F} = \langle x \cos y, y \rangle$ in moving a particle along the parabola $y = 2x^2$ from the point $(1, 2)$ to the point $(2, 8)$.

$\vec{F}$ is not conservative (why?)

therefore we must parameterize the curve.

let $x=t$, then $y=2t^2$, $1 \leq t \leq 2$

$$\vec{r}(t) = \langle t, 2t^2 \rangle, \quad \vec{r}'(t) = \langle 1, 4t \rangle$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \langle t \cos(2t^2), 2t^2 \rangle \cdot \langle 1, 4t \rangle dt$$

$$\int_1^2 \left(\underbrace{t \cos(2t^2)}_{u\text{-sub}} + 8t^3 \right) dt$$

$u\text{-sub}$

$$u = 2t^2$$

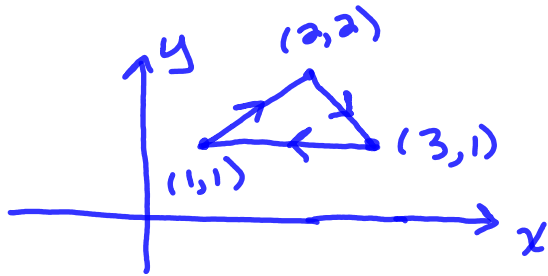
$$du = 4t dt$$

$$= \left(\frac{1}{4} \sin(2t^2) + 2t^4 \right) \Big|_1^2$$

$$= \frac{1}{4} \sin(8) + 32 - \left(\frac{1}{4} \sin(2) + 2 \right)$$

$$= \boxed{\frac{1}{4} \sin(8) - \frac{1}{4} \sin(2) + 30}$$

Problem 7. A particle is moving along a triangular path. The particle starts at the point $(1, 1)$, then to the point $(2, 2)$, then from $(2, 2)$ to the point $(3, 1)$, then back to the point $(1, 1)$. Find the work done on this particle by the force field $\mathbf{F} = \langle x + 1, y - 2x \rangle$.



This is a closed curve oriented clockwise. we can use Green's theorem as follows:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA, \quad P = x + 1 \\ Q = y - 2x$$

$$= - \iint_D (-2 - 0) dA$$

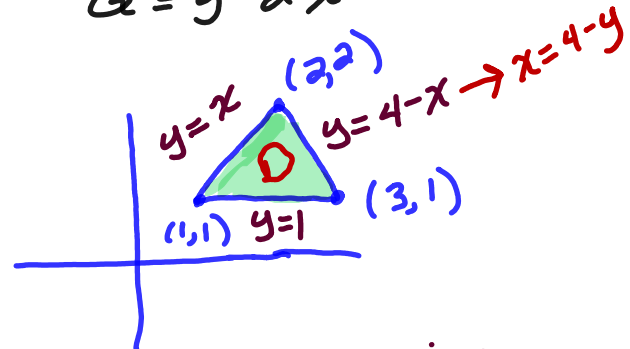
$$= \iint_D 2 dA$$

$$\int_1^2 \int_y^{4-y} 2 dx dy$$

$$\int_1^2 \left(2x \Big|_{x=y}^{x=4-y} \right) dy$$

$$\int_1^2 2(4-y-y) dy$$

$$\int_1^2 (8-4y) dy = \boxed{2}$$



To prevent having two integrals, choose $dA = dx dy$

$$D: 1 \leq y \leq 2$$

$$y \leq x \leq 4-y$$

set-up for $dA = dy dx$

For $1 \leq x \leq 2$, $1 \leq y \leq x$

For $2 \leq x \leq 3$, $1 \leq y \leq 4-x$

$$\int_1^2 \int_1^x 2 dy dx + \int_2^3 \int_1^{4-x} 2 dy dx$$

Problem 8. Evaluate $\int_C x dz + y dx + (xz) dy$, where C is parameterized by $\mathbf{r}(t) = \langle t^2, t^3, 2t \rangle$, $0 \leq t \leq 1$.

$$x = t^2$$
$$dx = 2t dt$$

$$y = t^3$$
$$dy = 3t^2 dt$$

$$z = 2t$$
$$dz = 2 dt$$

$$\int_C x dz + y dx + (xz) dy = \int_0^1 t^2 \cdot 2 dt + t^3 \cdot 2t dt + (2t^3) 3t^2 dt$$

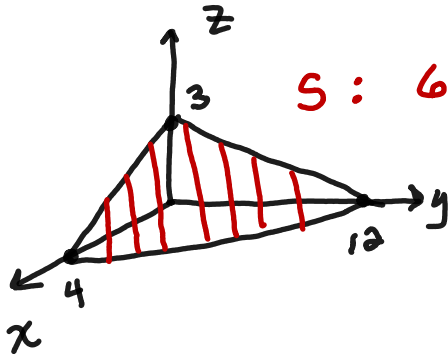
$$= \int_0^1 (2t^2 + 2t^4 + 6t^5) dt$$

$$= \left(\frac{2t^3}{3} + \frac{2t^5}{5} + t^6 \right) \Big|_0^1$$

$$= \left(\frac{2}{3} + \frac{2}{5} + 1 \right) \Big|_0^1$$

$$= \boxed{\frac{31}{15}}$$

Problem 9. Consider the part of the plane $6x + 2y + 8z = 24$ that lies in the first octant. Set up but **do not evaluate** a double integral that gives the surface area of this plane in the order $dA = dzdx$.

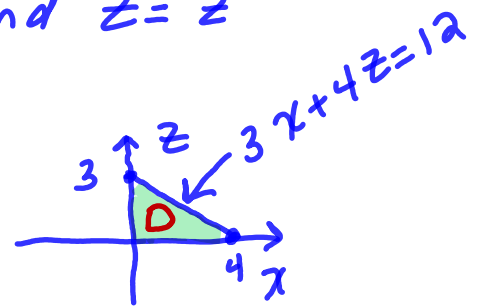


$$S: 6x + 2y + 8z = 24.$$

By solving this plane for y and choosing the parameters to be x and z , we will then be integrating in the xz -plane:

Let $x = x$, $y = \frac{24 - 6x - 8z}{2}$, and $z = z$

$$\vec{r}(x, z) = \langle x, 12 - 3x - 4z, z \rangle$$

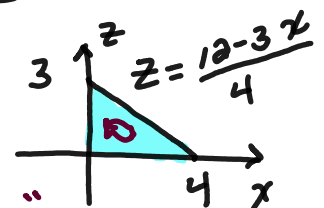


$$A(S) = \iint_D |\vec{r}_x \times \vec{r}_z| dA$$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 0 \\ 0 & -4 & 1 \end{vmatrix} = \langle -3, -1, -4 \rangle$$

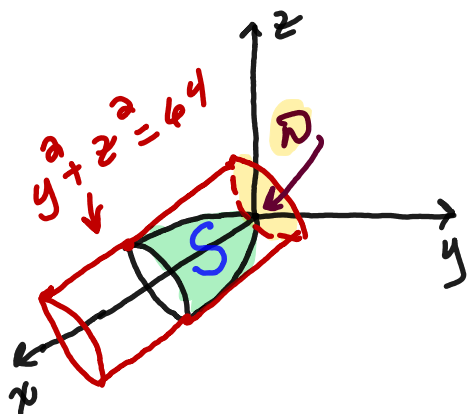
$A(S) = \iint_D \sqrt{26} dA$. Now, we were told to choose $dA = dz dx$

$$A(S) = \int_0^4 \int_0^{\frac{12-3x}{4}} \sqrt{26} dz dx$$



D as "Type I":
 $0 \leq x \leq 4, 0 \leq z \leq \frac{12-3x}{4}$

Problem 10. Find the surface area of the part of the paraboloid $x = 4y^2 + 4z^2$ that lies inside the cylinder $y^2 + z^2 = 64$.



parameterize S :

$$\text{let } y = y, z = z, x = 4y^2 + 4z^2$$

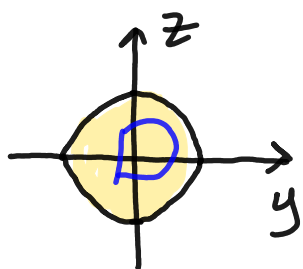
$$\vec{r}(y, z) = \langle 4y^2 + 4z^2, y, z \rangle, y^2 + z^2 \leq 64$$

$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8y & 1 & 0 \\ 8z & 0 & 1 \end{vmatrix}$$

$$= \langle 1, -8y, -8z \rangle$$

$$A(S) = \iint_{y^2 + z^2 \leq 64} |\vec{r}_y \times \vec{r}_z| dA = \iint_{y^2 + z^2 \leq 64} \sqrt{1 + 64y^2 + 64z^2} dA$$

D is polar in the yz -plane:



$$D : 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 8$$

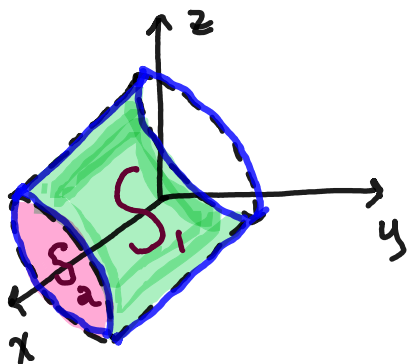
$$A(S) = \int_0^{2\pi} \int_0^8 \sqrt{1 + 64r^2} r dr d\theta$$

u -sub, $u = 1 + 64r^2$
 $du = 128r dr$
 sorry for huge
 number
 ↓

$$A(S) = \int_0^{2\pi} d\theta \int_0^8 r \sqrt{1 + 64r^2} dr$$

$$= \theta \Big|_0^{2\pi} \frac{1}{128} \frac{2}{3} (1 + 64r^2)^{3/2} \Big|_0^8 = (2\pi) \frac{1}{192} \left((4097)^{3/2} - 1 \right)$$

Problem 11. Consider the surface S that is the part of the cylinder $y^2 + z^2 = 9$, $0 \leq x \leq 2$, including the disk $x = 2$. Find a parameterization of S .



S_1 : the part of the cylinder

$$y^2 + z^2 = 9, \quad 0 \leq x \leq 2;$$

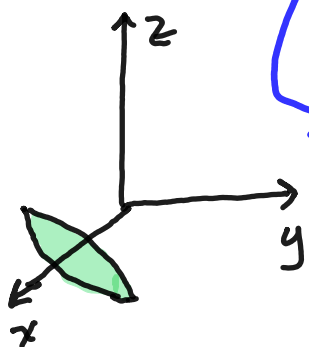
since we are
"on" the cylinder,

$$\vec{r}_1(\theta, x) = \langle x, 3\cos\theta, 3\sin\theta \rangle,$$

where $0 \leq x \leq 2$ and $0 \leq \theta \leq 2\pi$

$$r = 3$$

S_2 : The disk in the plane $x = 2$
that is inside the cylinder $y^2 + z^2 = 9$:



$\left\{ \begin{array}{l} S_2 \text{ is the disk } y^2 + z^2 \leq 9 \\ \text{that lies in the } x = 2 \text{ plane.} \end{array} \right.$

$$\vec{r}_2(r, \theta) = \langle 2, r\cos\theta, r\sin\theta \rangle$$

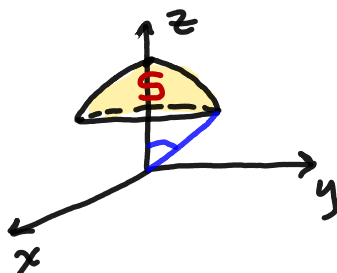
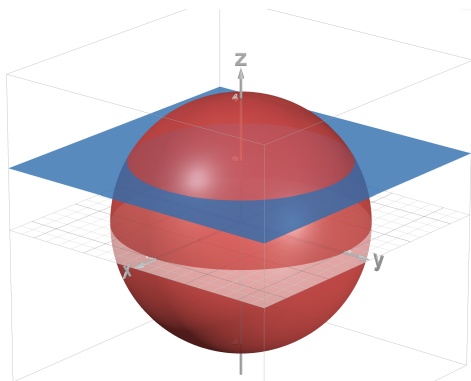
Here, since
we are inside
a circle,
 $r = \text{variable}$

$$0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi$$

S is the union of S_1 and S_2

Problem 12. Evaluate $\iint_S z \, dS$ where S is the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the plane $z = 2$.

Recall: A sphere of radius ρ is parameterized by $\mathbf{r}(\phi, \theta) = \langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi) \rangle$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. Furthermore, $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \rho^2 \sin(\phi)$. NOTE: When parameterizing a sphere, ρ is the radius of the sphere, which is a constant!!



$\rho = 4$
 $\boxed{0 \leq \theta \leq 2\pi}$
 To find the upper bound on ϕ :

$$z = 4 \cos \phi$$

$$2 = 4 \cos \phi$$

$$\phi = \frac{\pi}{3}$$

$$\boxed{0 \leq \phi \leq \frac{\pi}{3}}$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 16 \sin \phi$$

$$\iint_S z \, dS = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \underbrace{4 \cos \phi}_{z = \rho \cos \phi, \rho=4} \underbrace{16 \sin \phi}_{\rho^2 \sin \phi, \rho=4} d\phi d\theta$$

$$= 64 \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \cos \phi \sin \phi d\phi d\theta$$

Fubini!

$$= 64 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \cos \phi \sin \phi d\phi$$

$$= 64 \cdot \theta \Big|_0^{2\pi} \frac{\sin^2 \phi}{2} \Big|_0^{\frac{\pi}{3}}$$

$$= 64(\cancel{2\pi}) \cdot \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right)^2 = 64\pi \left(\frac{3}{4} \right) = \boxed{48\pi}$$

u-sub, $u = \sin \phi$
 $du = \cos \phi d\phi$
 $\int u du = \frac{u^2}{2} = \frac{\sin^2 \phi}{2}$

Problem 13. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle xy, 4x^2, yz \rangle$ and S is the surface $z = xe^y$, $0 \leq x \leq 1$, $0 \leq y \leq 1$, with upward orientation.

let $\vec{r}(x, y) = \langle x, y, xe^y \rangle$, $0 \leq x \leq 1, 0 \leq y \leq 1$ D

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & e^y \\ 0 & 1 & xe^y \end{vmatrix} = \langle -e^y, -xe^y, 1 \rangle$ ↙ z > 0, upward orientation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\vec{r}(x, y)) \cdot (\vec{r}_x \times \vec{r}_y) dA$$

$$= \iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} \langle xy, 4x^2, y \cdot xe^y \rangle \cdot \langle -e^y, -xe^y, 1 \rangle dA$$

$$= \iint_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1}} (-xy \cancel{e^y} - 4x^3 e^y + y \cancel{x} e^y) dA$$

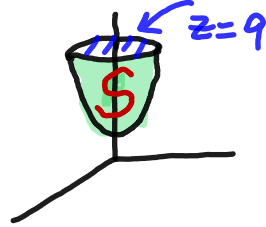
$$= \int_0^1 \int_0^1 (-4x^3 e^y) dy dx \quad \text{Fubini!!}$$

$$= -4 \int_0^1 x^3 dx \int_0^1 e^y dy = -4 \cdot \frac{x^4}{4} \Big|_0^1 e^y \Big|_0^1 = \boxed{-(e-1)}$$

Problem 14. Set up but do not evaluate a double integral that gives the flux of $\mathbf{F} = \langle z - 3, x, y \rangle$ across S , where S is the part of the paraboloid $z = x^2 + y^2 + 3$ that is below the plane $z = 9$.

Use the **positive** (outward) **orientation**.

$$\text{Flux } \mathbf{F} = \iint_S \mathbf{F} \cdot d\mathbf{s}$$



$$z = x^2 + y^2 + 3$$

$$6 = x^2 + y^2$$

$$\vec{r}(x, y) = \langle x, y, x^2 + y^2 + 3 \rangle, \quad x^2 + y^2 \leq 6$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = \langle -2x, -2y, 1 \rangle$$

$z > 0$ ✓
positive
orientation

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_{x^2 + y^2 \leq 6} \langle x^2 + y^2 + 3 - 3, x, y \rangle \cdot \langle -2x, -2y, 1 \rangle dA$$

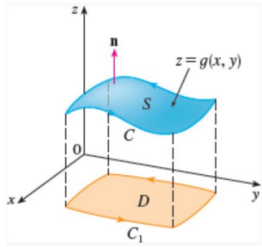
$$= \iint_{x^2 + y^2 \leq 6} (-2x(x^2 + y^2) - 2xy + y) dA$$

convert
to polar

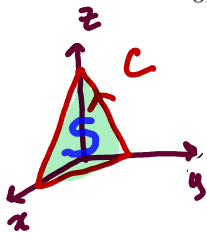
$$= \int_0^{2\pi} \int_0^{\sqrt{6}} (-2r \cos \theta (r^2) - 2r^2 \cos \theta \sin \theta + r \sin \theta) r dr d\theta$$

Stokes' Theorem: Let S be an oriented piecewise-smooth surface parameterized by $\mathbf{r}(u, v)$, $u, v \in D$, that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive (counterclockwise) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains the surface S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$



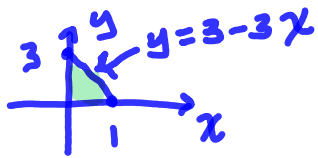
Problem 15. Use Stokes' Theorem to set up but **not evaluate** $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle yz, 2xy, 4xz \rangle$, and where C is the boundary curve of the part of the plane $3x + y + z = 3$ in the first octant, oriented counterclockwise when looking from above.



$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xy & 4xz \end{vmatrix} = \langle 0, y-4z, 2y-z \rangle$$

Stokes: $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where C = boundary curve of S .

since C is ccw, S is the positively oriented plane $3x + y + z = 3$ in first octant. If we parameterize S as $\vec{r}(x, y) = \langle x, y, 3-3x-y \rangle$, then D will be in the xy -plane:



$$D: 0 \leq x \leq 1 \\ 0 \leq y \leq 3-3x$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{vmatrix} = \langle 3, 1, 1 \rangle$$

$z > 0$, positive orientation ✓

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D \text{curl } \mathbf{F}(\vec{r}(x, y)) \cdot (\vec{r}_x \times \vec{r}_y) dA$$

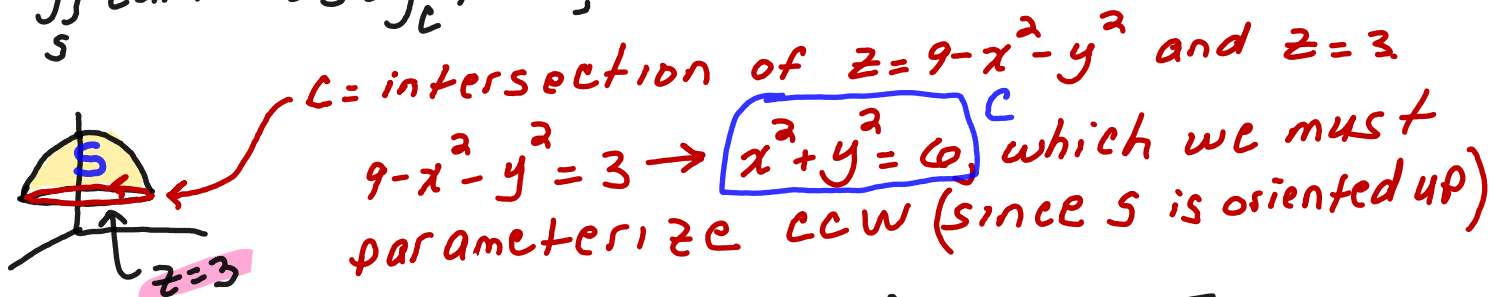
$$\text{curl } \mathbf{F} = \langle 0, y-4z, 2y-z \rangle$$

$$= \iint_D \langle 0, \underbrace{y-4(3-3x-y)}_{y-4z}, \underbrace{2y-(3-3x-y)}_{2y-z} \rangle \cdot \langle 3, 1, 1 \rangle dA$$

$$= \int_0^1 \int_0^{3-3x} \left(y-4(3-3x-y) + 2y-(3-3x-y) \right) dy dx$$

Problem 16. Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x^2 \sin(z-3), y^2, xy \rangle$, and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 3$, oriented upward.

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}, \text{ where } C = \text{boundary of } S.$$



$$C: \vec{r}(\theta) = \langle \sqrt{6} \cos \theta, \sqrt{6} \sin \theta, 3 \rangle, 0 \leq \theta \leq 2\pi$$

$$\vec{r}'(\theta) = \langle -\sqrt{6} \sin \theta, \sqrt{6} \cos \theta, 0 \rangle \quad \vec{F} = \langle x^2 \sin(z-3), y^2, xy \rangle$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\vec{r}(\theta)) \cdot \vec{r}'(\theta) d\theta$$

$$= \int_0^{2\pi} \langle 6 \cos^2 \theta \sin(\theta), 6 \sin^2 \theta, 6 \cos \theta \sin \theta \rangle \cdot \langle -\sqrt{6} \sin \theta, \sqrt{6} \cos \theta, 0 \rangle d\theta$$

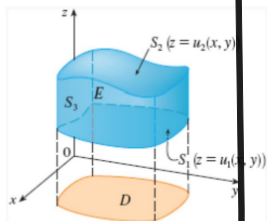
$$= \int_0^{2\pi} 6\sqrt{6} \sin^2 \theta \cos \theta d\theta$$

$$u = \sin \theta \quad du = \cos \theta d\theta \quad \int u^2 du = \frac{u^3}{3}$$

$$= 6\sqrt{6} \frac{\sin^3 \theta}{3} \Big|_0^{2\pi}$$

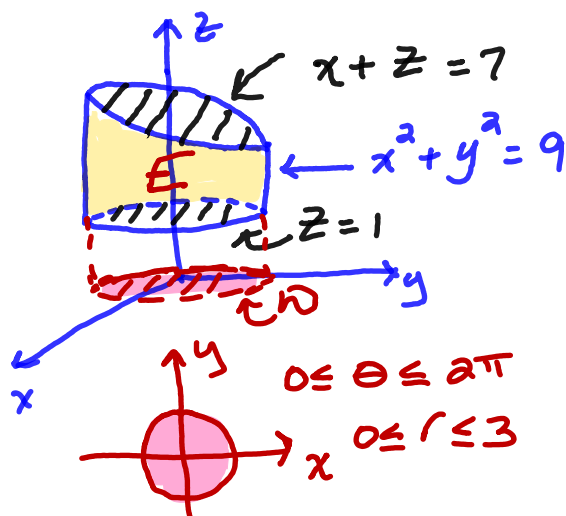
$$= 0$$

The Divergence Theorem: Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E . Then $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV$.



Problem 17. Using the The Divergence Theorem, set up but do not evaluate a triple integral used to find the flux of $\mathbf{F} = \langle ye^{z^2}, ze^x, 2z + 8 \rangle$ across S , where S is the surface of the solid bounded by the plane $x + z = 7$, the cylinder $x^2 + y^2 = 9$, and the plane $z = 1$.

$$\text{Flux } \mathbf{F} = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV = \iiint_E 2 dV$$



Define E in cylindrical coordinates

$$1 \leq z \leq 7 - x, \quad x^2 + y^2 \leq 9$$

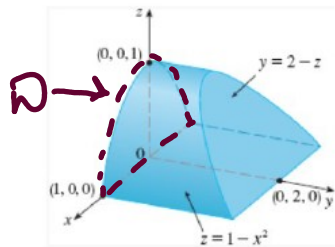
$$1 \leq z \leq 7 - r \cos \theta$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 3$$

$$\iiint_E 2 dV = \int_0^{2\pi} \int_0^3 \int_1^{7-r\cos\theta} 2 \cdot dz r dr d\theta$$

Problem 18. Using The Divergence Theorem, set up but do not evaluate a triple integral used to find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. Assume S is positively orientated (see picture below).



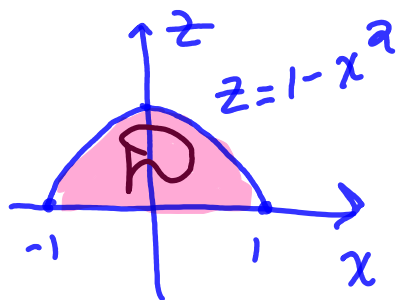
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \cdot dV$$

$$= \iiint_E (y + 2y) dV$$

$$= \iiint_E 3y dV$$

If we choose
 $dV = dy dz$, then
 $0 \leq y \leq 2 - z$,

and D will be in
the xz -plane



$$-1 \leq x \leq 1$$

$$0 \leq z \leq 1 - x^2$$

$$E: \begin{aligned} 0 &\leq y \leq 2 - z \\ 0 &\leq z \leq 1 - x^2 \\ -1 &\leq x \leq 1 \end{aligned}$$

$$\iiint_E 3y dV = \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 2y dy dz dx$$

Problem 19. Don't forget to review section 14.7, local and absolute extrema. Find all local extrema and/or saddle points if $f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$.

$$\text{CP: } f_x(x, y) = -6x^2 - 12x + 48 = -6(x^2 + 2x - 8) = -6(x+4)(x-2)$$

$$f_y(x, y) = 3y^2 - 12y = 3y(y-4)$$

$$f_x = 0 \text{ when } x = -4, x = 2$$

$$f_y = 0 \text{ when } y = 0, y = 4$$

$$\text{CP: } (-4, 0), (-4, 4), (2, 0), (2, 4)$$

Test these critical points using the second derivative

$$\text{test: } D = f_{xx}f_{yy} - (f_{xy})^2 = (-12x-12)(6y-12) - 0$$

CP	$D = (-12x-12)(6y-12)$	$f_{xx} = -12x-12$	conclusion: $(a, b, f(a, b)) =$
$(-4, 0)$	$D(-4, 0) < 0$	N/A	saddle pt $(-4, 0, -44)$
$(-4, 4)$	$D(-4, 4) > 0$	$f_{xx}(-4, 4) > 0$	local min $(-4, 4, -172)$ or $z = -172$
$(2, 0)$	$D(2, 0) > 0$	$f_{xx}(2, 0) < 0$	local max $(2, 0, 76)$ or $z = 76$
$(2, 4)$	$D(2, 4) < 0$	N/A	saddle pt $(2, 4, 44)$