Week in Review
Math 152

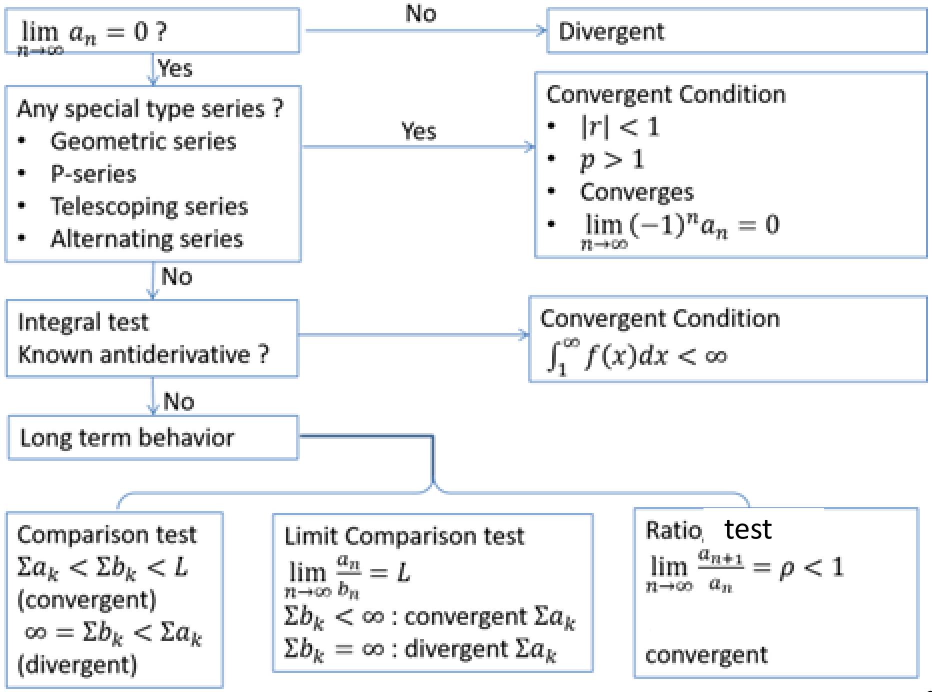
Week 11

Alternating series

Absolute Convergence and the Ratio Test

Power series

Alternating series



+/- terms but Not alternating
⇒ Absolute convergence

Determine if the alternating series converges or diverges

$$\sum_{i=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$$

$$\sum_{i=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+1}$$

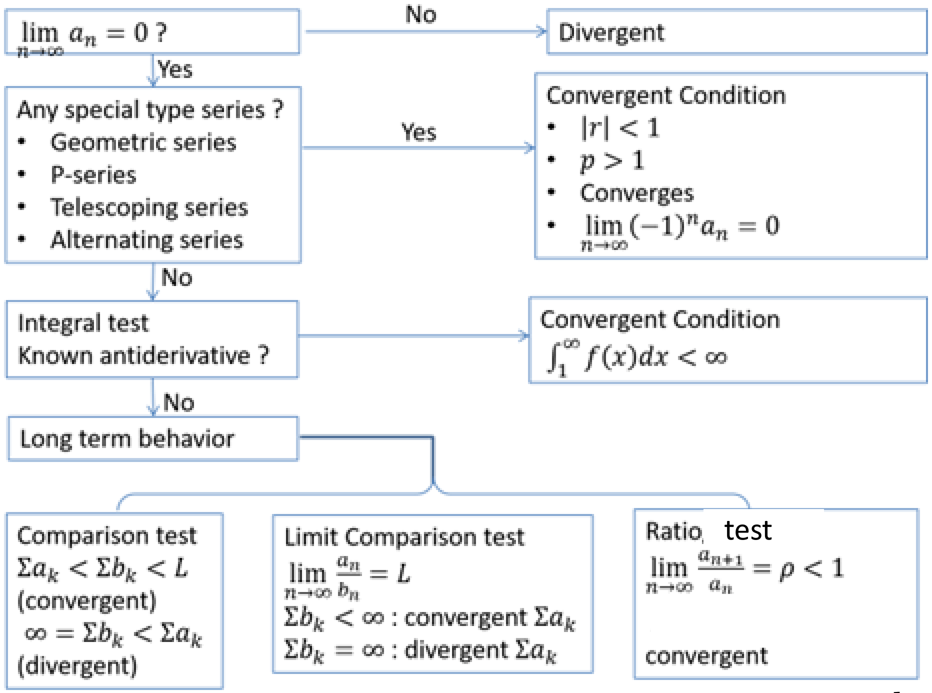
$$\lim_{n\to\infty}\frac{n}{n^2+1} = 0$$

$$\Rightarrow \sum_{i=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1} \text{ converges}$$

$$\lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$$

$$\Rightarrow \sum_{i=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+1} \text{ diverges}$$

Alternating series



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Determine if the alternating series converges or diverges

$$\sum_{i=1}^{\infty} (-1)^n \frac{\ln n}{n}$$

$$\sum_{i=1}^{\infty} (-1)^{n+1} \ln\left(1 + \frac{1}{n}\right)$$

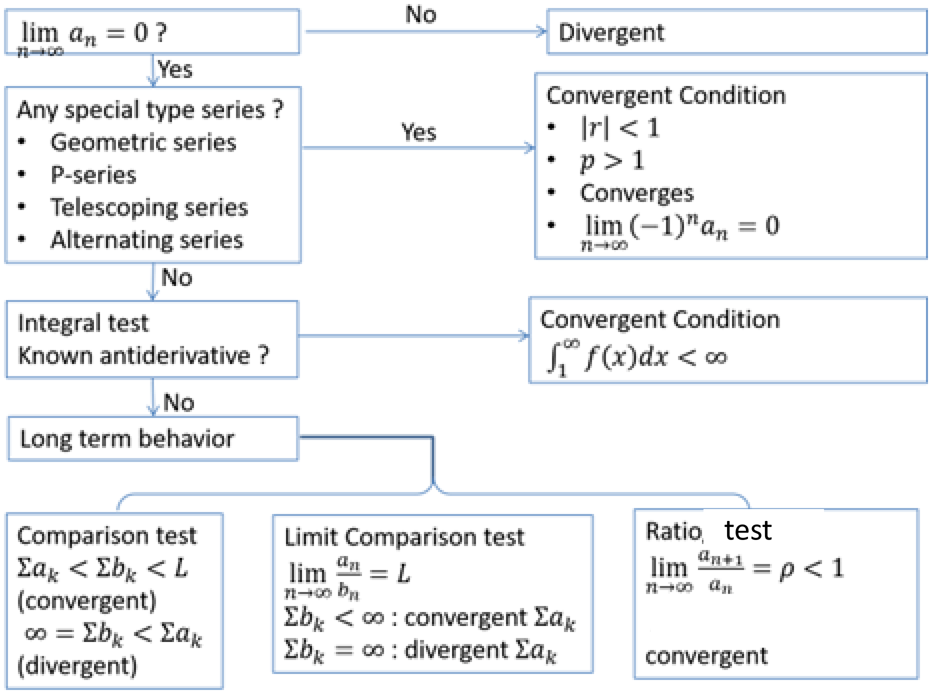
$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$\Rightarrow \sum_{i=1}^{\infty} (-1)^n \frac{\ln n}{n} \text{ converges}$$

$$\lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = 0$$

$$\Rightarrow \sum_{i=1}^{\infty} (-1)^{n+1} \ln\left(1 + \frac{1}{n}\right) \text{ converges}$$

Alternating series



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Determine if the alternating series converges or diverges

$$\sum_{i=1}^{\infty} (-1)^n \frac{\sqrt{n}+1}{n+1}$$

$$\sum_{i=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n} - \frac{1}{n^2}\right)$$

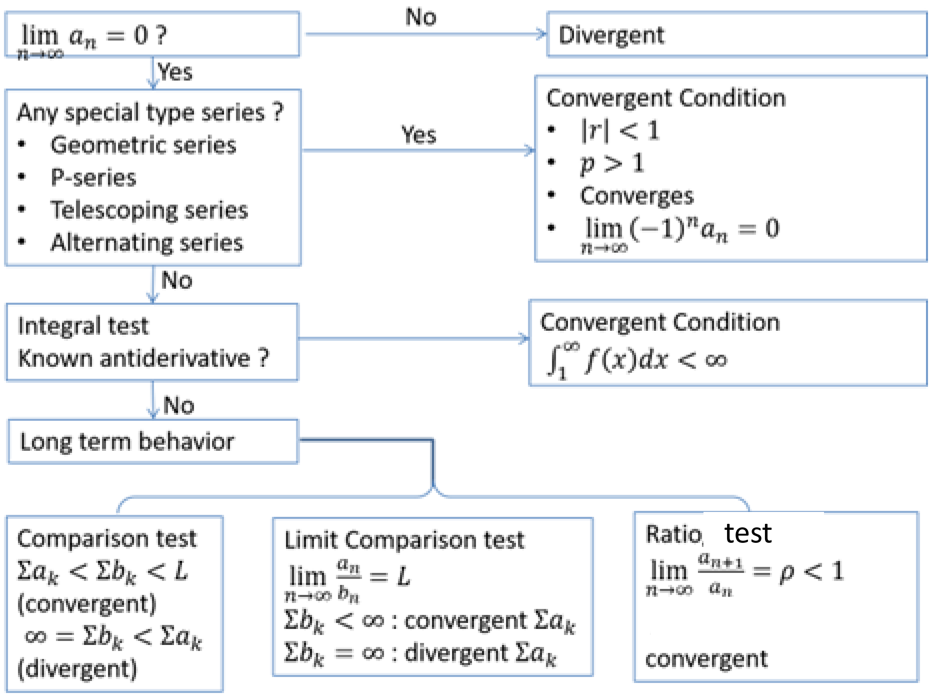
$$\lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = 0$$

$\Rightarrow \sum_{i=1}^{\infty} (-1)^n \frac{\sqrt{n}+1}{n+1}$ converges

$$\lim_{n\to\infty} \frac{n-1}{n^2} = 0$$

$\Rightarrow \sum_{i=1}^{\infty} (-1)^{n+1} \ln\left(1 + \frac{1}{n}\right)$ converges

Alternating series



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

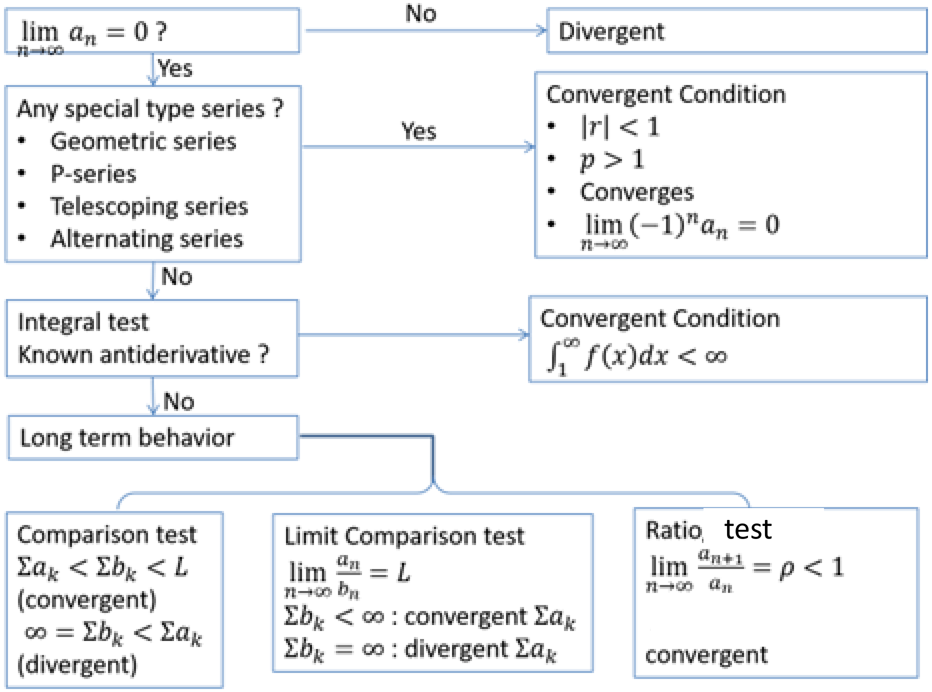
Estimate the magnitude of the error $|S - S_n|$

$\sum_{i=1}^{\infty} \frac{(-1)^n}{n}$

$\sum_{i=1}^{\infty} (-r)^n$ with $|r| < 1$

- $|S - S_n| < |a_{n+1}| = \frac{1}{n}$
- $|S - S_n| < |a_{n+1}| = |r^{n+1}|$

Absolute Convergence



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

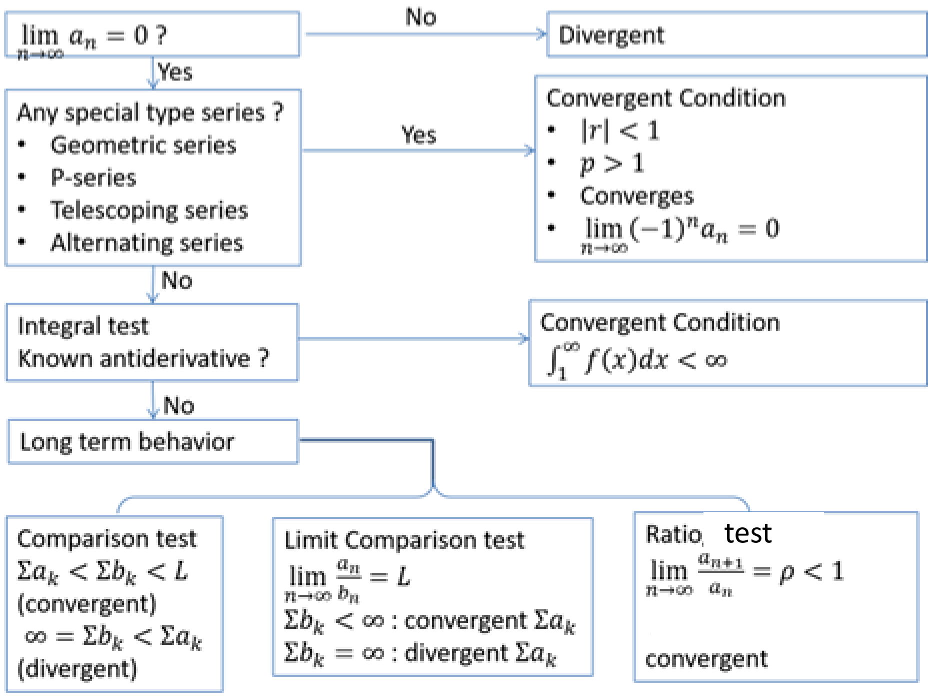
$$\sum_{i=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

$$\sum_{i=1}^{\infty} \frac{(-1)^n}{\ln(n+2)}$$

- $|S - S_n| < |a_{n+1}| = \frac{n+1}{(n+1)^2+1} < \frac{1}{1000}$
- $1000(n+1) < (n+1)^2 + 1$
- $k^2 - 1000k + 1 > 0$ w/ $k = n + 1$
- $n + 1 > 500 + \sqrt{500^2 + 1} = 1000.001$
- $n > 999.001 \quad \Rightarrow n = 1000$

- $|S - S_n| < |a_{n+1}| = \frac{1}{\ln(n+3)} < \frac{1}{1000}$
- $\ln(n+3) > 10^3 \quad \Rightarrow n + 3 > e^{10^3}$
- $n > e^{10^3} - 3$

Absolute Convergence



+/- terms but Not alternating
⇒ Absolute convergence

Determine which of the converge absolutely, converge, and diverge?

$\sum_{i=1}^{\infty}(-1)^n\,(0.1)^n$

$\sum_{i=1}^{\infty}(-1)^{n+1}\,\frac{(0.1)^n}{n}$

- $\lim_{n\to\infty}(0.1)^n = 0$ (also geometric series)

$\Rightarrow \sum_{i=1}^{\infty}(-1)^n\,(0.1)^n$ converges

- $\sum_{i=1}^{\infty}(0.1)^n$ converges (converge absolutely)

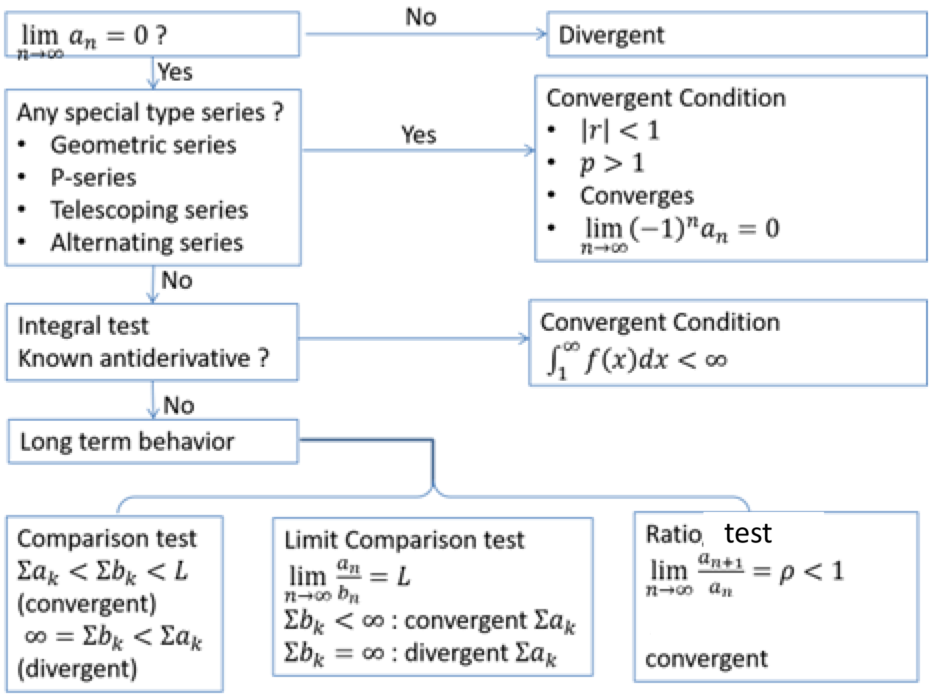
- $\lim_{n\to\infty}\frac{(0.1)^n}{n} = 0$

$\Rightarrow \sum_{i=1}^{\infty}(-1)^{n+1}\,\frac{(0.1)^n}{n}$ converges

- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(0.1)^{n+1}(n)}{(0.1)^n(n+1)}\right| = 0.1$

$\Rightarrow \sum_{i=1}^{\infty}(-1)^{n+1}\,\frac{(0.1)^n}{n}$ converge absolutely

Absolute Convergence



+/- terms but Not alternating
⇒ Absolute convergence

Determine which of the converge absolutely, converge, and diverge?

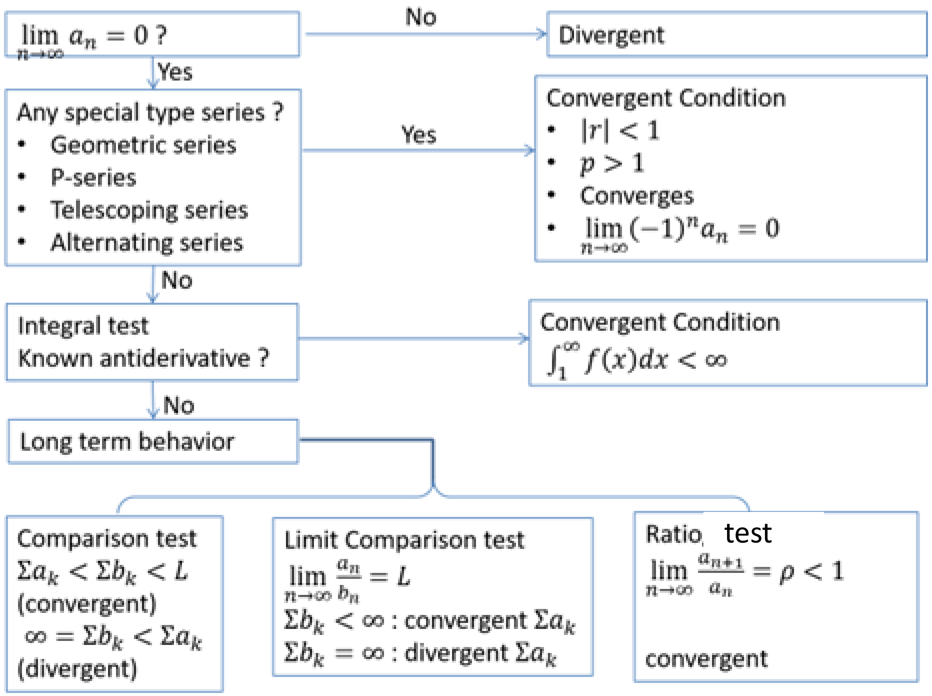
$\sum_{i=1}^{\infty} \frac{(-1)^n n^3}{3^n}$

$\sum_{i=1}^{\infty} \frac{(-1)^n \sqrt{n}}{3^n}$

$\sum_{i=1}^{\infty} \frac{(-1)^n 3^n}{2^{2n}}$

- $\sum_{i=1}^{\infty} \frac{(-1)^n n^3}{3^n}$ converges absolutely
  - $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3 3^n}{3^{n+1} n^3}\right| = \frac{1}{3}$
- $\sum_{i=1}^{\infty} \frac{(-1)^n \sqrt{n}}{3^n}$ converges absolutely
  - $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}\, 3^n}{3^{n+1}\sqrt{n}}\right| = \frac{1}{3}$
- $\sum_{i=1}^{\infty} \frac{(-1)^n 3^n}{2^{2n}}$ converges absolutely
  - Harmonic series

Absolute Convergence



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Determine which of the converge absolutely, converge, and diverge?

$\sum_{i=2}^{\infty}(-1)^n \frac{1}{n \ln n}$

$\sum_{i=2}^{\infty}(-1)^n \frac{1}{n - \ln n}$

- $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$

$\Rightarrow \sum_{i=2}^{\infty}(-1)^n \frac{1}{n \ln n}$ converges

- $\sum_{i=1}^{\infty} \frac{1}{n \ln n}$ diverges (integral test)

$\Rightarrow \sum_{i=2}^{\infty}(-1)^n \frac{1}{n \ln n}$ converges conditionally

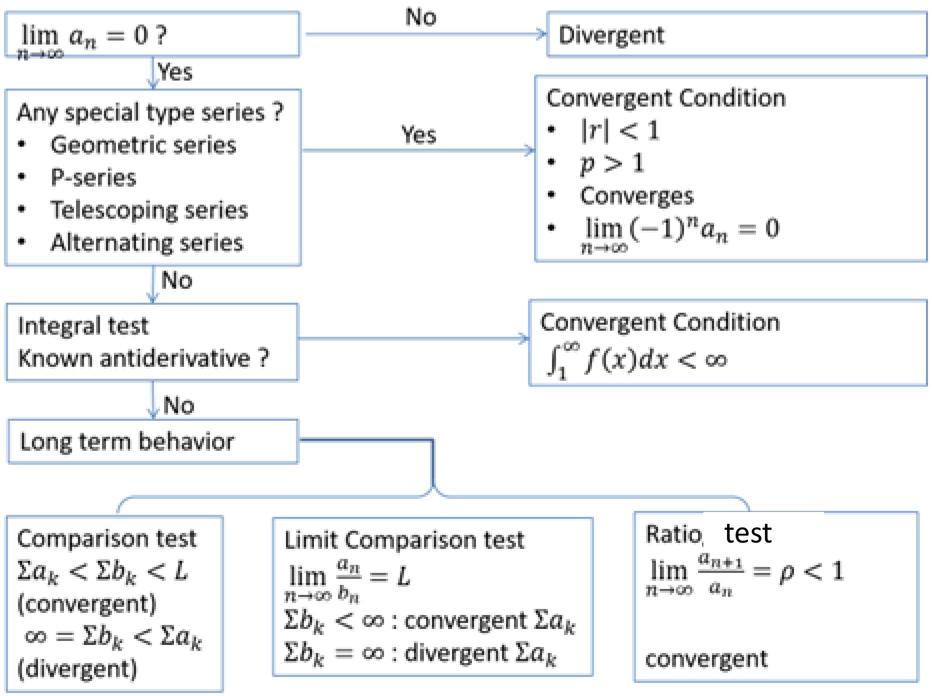
- $\lim_{n\to\infty} \frac{1}{n - \ln n} = 0$

$\Rightarrow \sum_{i=2}^{\infty}(-1)^n \frac{1}{n - \ln n}$ converges

- $\lim_{n\to\infty} \left| \frac{\frac{1}{n-\ln n}}{\frac{1}{n}} \right| = 1$ and $\lim_{n\to\infty} \frac{1}{n} = \infty$

$\Rightarrow \sum_{i=2}^{\infty}(-1)^n \frac{1}{n - \ln n}$ converges conditionally

Absolute Convergence



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Determine which of the converge absolutely, converge, and diverge?

$\sum_{i=2}^{\infty}(-1)^n \frac{1}{n(\ln n)^2}$

$\sum_{i=2}^{\infty}(-1)^n \frac{1}{(\ln n)^2}$

- $\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$

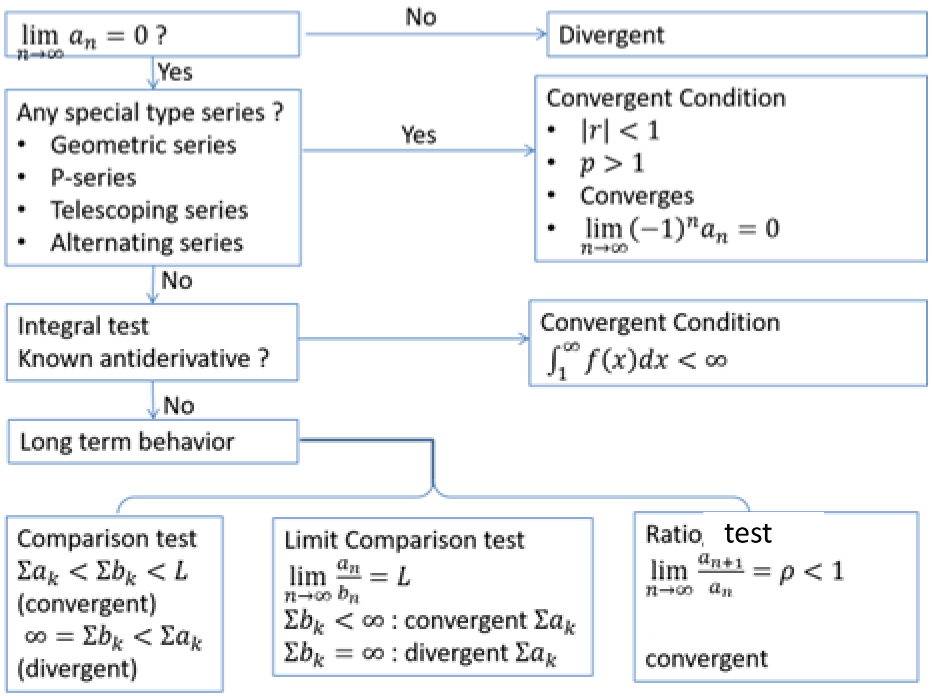
$\Rightarrow \sum_{i=2}^{\infty}(-1)^n \frac{1}{n(\ln n)^2}$ converges

- $\sum_{i=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges (integral test)

$\Rightarrow \sum_{i=2}^{\infty}(-1)^n \frac{1}{n(\ln n)^2}$ converges absolutely

- $\lim_{n\to\infty} \frac{(-1)^n}{(\ln n)^2} = 0 \Rightarrow \sum_{i=2}^{\infty}(-1)^n \frac{1}{(\ln n)^2}$ converges
- $\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0$ and $\sum_{i=2}^{\infty} \frac{1}{n} = \infty \Rightarrow \sum_{i=2}^{\infty} \frac{1}{(\ln n)^2} = \infty$
- $\sum_{i=2}^{\infty}(-1)^n \frac{1}{(\ln n)^2}$ converges conditionally

Absolute Convergence



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

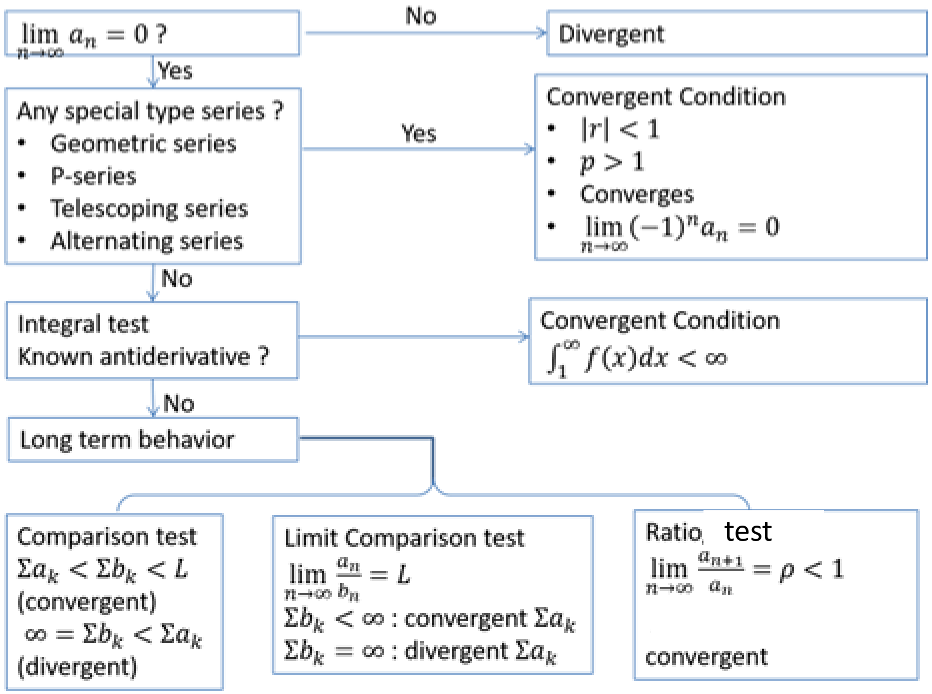
Determine which of the converge absolutely, converge, and diverge?

$\sum_{i=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

$\sum_{i=1}^{\infty} \frac{\cos n\pi}{n}$

- $\sum_{i=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}} = \sum_{i=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$
  - Alternating p series (p=3/2)
  - Converges absolutely
- $\sum_{i=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{i=1}^{\infty} \frac{(-1)^n}{n}$
  - Alternating harmonic series
  - Converges conditionally

Absolute Convergence



+/- terms but Not alternating
⇒ Absolute convergence

Determine which of the converge absolutely, converge, and diverge?

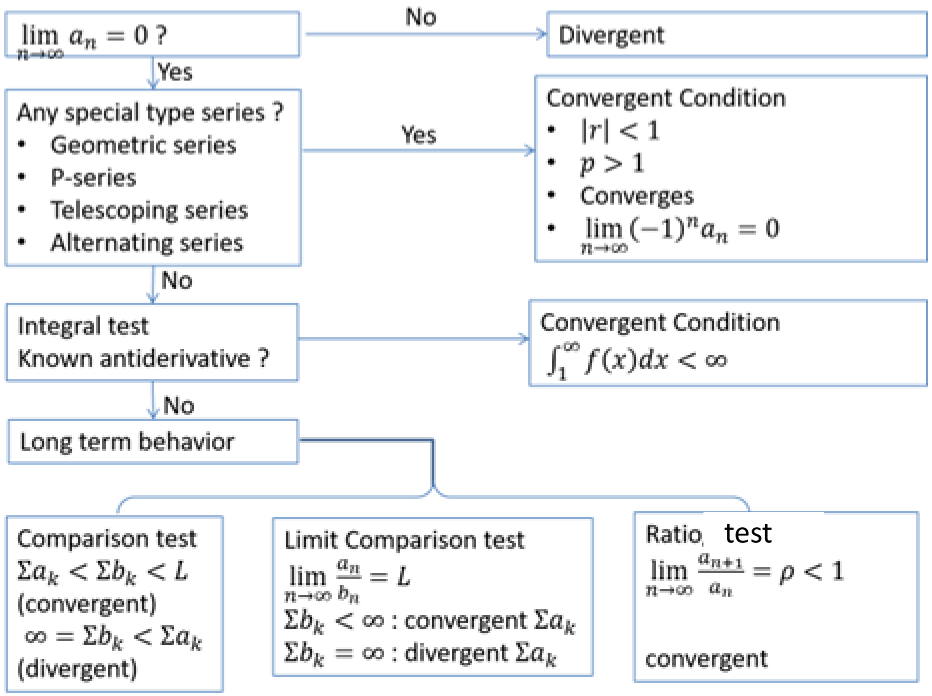
$\sum_{i=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$

$\sum_{i=1}^{\infty}(-1)^n\left(\sqrt{n^2+n}-n\right)$

- $\sum_{i=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)$
  - $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$ Converges
  - $\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{2\sqrt{n}}} = 1$ w/ $\sum_{i=1}^{\infty}\frac{1}{2\sqrt{n}} = \infty$
  - Converges conditionally

- $\sum_{i=1}^{\infty}(-1)^n\left(\frac{n}{\sqrt{n^2+1}+n}\right)$
  - $\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n} = \frac{1}{2}$
  - Divergent

Absolute Convergence



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

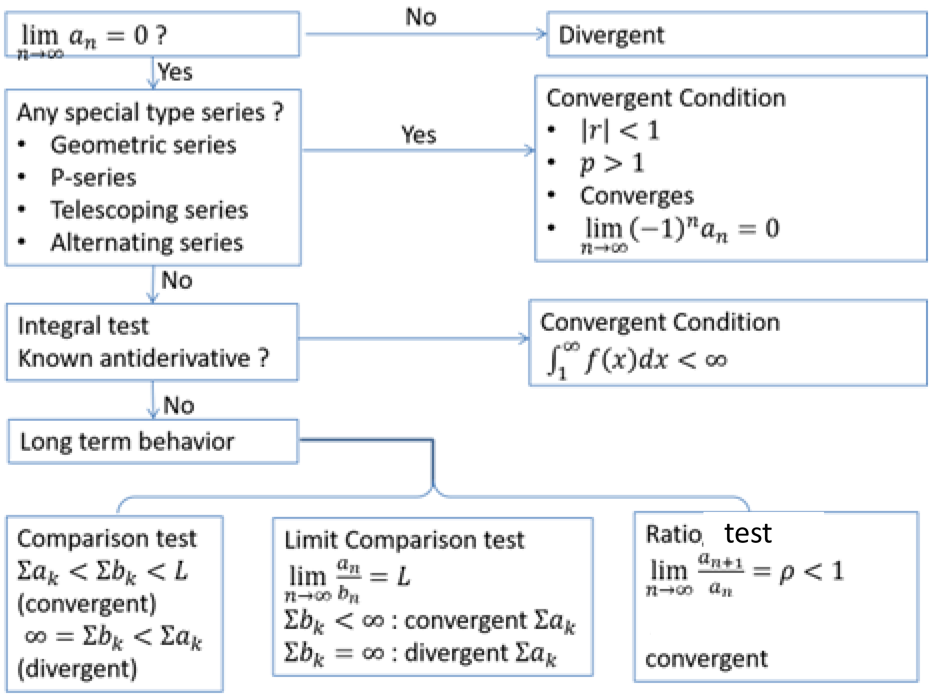
Determine which of the converge absolutely, converge, and diverge?

$$\sum_{i=1}^{\infty} \frac{(-1)^n}{(\sqrt{n+1}+\sqrt{n})}$$

$$\frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14} + \cdots$$

- $\sum_{i=1}^{\infty} \frac{(-1)^n}{(\sqrt{n+1}+\sqrt{n})}$ converges conditionally
  - $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$ Converges
  - $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{2\sqrt{n}}} = 1$ w/ $\sum_{i=1}^{\infty} \frac{1}{2\sqrt{n}} = \infty$
- $\sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{2(n+1)} \sim$ Alternating harmonic series
  - converges conditionally

Absolute Convergence



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Determine which of the converge absolutely, converge, and diverge?

$$\sum_{i=1}^{\infty} \frac{(-1)^n (n+1)^n}{(n)^n}$$

$$\sum_{i=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$$

$$\sum_{i=1}^{\infty} \frac{(-1)^n (n+1)^n}{(n)^n} = \sum_{i=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$: Diverges

$$\sum_{i=1}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$$

Absolute convergent

Absolute Convergence

Which of the following series is absolutely convergent by **the Ratio Test**?

(I) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$ (II) $\sum_{n=1}^{\infty} \frac{n^4(-2)^n}{n!}$ (III) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+4}$

(a) I and II only ← correct

(b) I only

(c) II only

(d) II and III only

(e) I, II, and III

(I) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3 3^n}{3^{n+1} n^3}\right| = \frac{1}{3} < 1$ (Absolute convergence)

(II) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4(-2)^{n+1} n!}{(n+1)!\, n^4(-2)^n}\right| = 0 < 1$ (Absolute convergence)

(III) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{(n+1)^3+4}\cdot\frac{n^3+4}{n}\right| = 1$ (Ratio test fails)

$\sum_{i=1}^{\infty} \frac{n}{n^3+4} < \sum_{i=1}^{\infty} \frac{1}{n^2} < \infty$ (Absolute convergence)

Convergence of power series

A power series about a, or just power series, is any series that can be written in the form,

$$S(x) = \sum_{n=1}^{\infty} c_n (x-a)^n \text{ where } a, c_n \in \mathbb{R}$$

- The c_n's are often called the coefficients of the series.
- A power series is that it is a function of x.
  - For different x, the power series may or may not converges
- There is a number R so that the power series will converge for, $|x-a| < R|$ and will diverge for $|x-a| > R$. This number, R is called the radius of convergence for the series
  - The series may or may not converge if $|x-a| = R$
- The interval of all x's, including the endpoints ($|x-a| = R$), for which the power series converges is called the interval of convergence of the series

- To find the radius of convergence, we apply ratio test for absolute convergence of the power series
- To find the interval of convergence, investigate the convergence at the endpoints $|x-a| = R$

Example: Plot the partial sums of $\sum_{n=1}^{\infty} x^n$

Convergence of power series

Determine the radius of convergence and interval of convergence for the following power series.

$\sum_{i=1}^{\infty} \frac{(-1)^n (x+1)^n}{4^n}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x+1)^{n+1} 4^n}{4^{n+1}(x+1)^n}\right| = \lim_{n\to\infty} \left|\frac{(x+1)}{4}\right| = \left|\frac{x+1}{4}\right| < 1$

$|x+1| < 4 \Rightarrow$ RoC: $R = 4$

- Endpoints: $x + 1 = \pm 4 \Rightarrow x = 3$, or $x = -5$
  - $@x = 3, S(x) = \sum_{i=1}^{\infty} \frac{(-1)^n 4^n}{4^n} = \sum_{i=1}^{\infty} (-1)^n$ (divergent)
  - $@x = 3, S(x) = \sum_{i=1}^{\infty} \frac{(-1)^n (-4)^n}{4^n} = \sum_{i=1}^{\infty} 1$ (divergent)
- IoC: $-5 < x < 3$

$\sum_{i=1}^{\infty} \frac{(x-1)^n}{n!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1} n!}{(n+1)!(x-1)^n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)}{n}\right| = 0 < 1$ for all x

$\Rightarrow$ RoC: $R = \infty$

- IoC: $-\infty < x < \infty$

$\sum_{i=1}^{\infty} n!\,(2x+1)^n$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{2x+1}\right| = \infty > 1$ for all $x \neq -\frac{1}{2}$

$\Rightarrow$ RoC: $R = 0$

- IoC: $x = -\frac{1}{2}$

Convergence of power series

Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(-2)^n x^2}{(n+1)!}$.

(a) 0

(b) ∞ ← correct

(c) $\frac{1}{2}$

(d) 1

(e) 2

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(-2)^{n+1}x^2}{(n+2)!} \cdot \frac{(n+1)!}{(-2)^n x^2}$$
$$= \lim_{n\to\infty} \frac{-2}{(n+2)} = 0 \text{ for all } x$$

Convergence of power series

The series $\sum_{n=1}^{\infty} c_n(x+1)^n$ converges when $x = -4$. Which of the following series is guaranteed to converge?

(I) $\sum_{n=1}^{\infty} c_n \cdot 0^n$ (II) $\sum_{n=1}^{\infty} c_n$ (III) $\sum_{n=1}^{\infty} c_n 2^n$ (IV) $\sum_{n=1}^{\infty} c_n 3^n$

(a) I and II only
(b) I, II, and III only $\leftarrow$ correct
(c) II and III only
(d) II, III, and IV only
(e) I, II, III and IV