# Math 308: Week-in-Review 10 (Exam 2 Review)

## 1 3.7-3.8: Mechanical Vibrations

### Review

**Standard Equation:** $mu'' + cu' + ku = F(t)$, where $m$ is mass, $c$ is damping coefficient, $k$ is spring constant, $u(t)$ is displacement, and $F(t)$ is the external force.

**Parameters:**

- $m$: Mass of the object.
- $k = \dfrac{F}{u}$ (Hooke's law).
- $c$: Linear damping $F_D = c\,v$ where $v$ is the velocity, $F_D$ is the damping force

**Solutions:**

- Homogeneous: $(u_h(t))$ when $F(t) = 0$:
- Overdamped: $u_h(t) = C_1e^{r_1t} + C_2e^{r_2t}$
- Critically damped: $u_h(t) = (C_1 + C_2t)e^{-\frac{c}{2m}t}$ at $c = 2\sqrt{mk}$
- Underdamped: $u_h(t) = e^{-\frac{c}{2m}t}(C\cos(\omega_d t) + D\sin(\omega_d t))$, where $\omega_d = \sqrt{\dfrac{k}{m} - \left(\dfrac{c}{2m}\right)^2}$
- Particular (steady state solution $u_p(t)$): Depends on $F(t)$.
  If $F(t) = F_0\cos(\omega t) \implies u_p(t) = A\cos(\omega t) + B\sin(\omega t)$
- Total: $u(t) = u_h(t) + u_p(t)$; transients decay, leaving steady-state $u_p(t)$.

**Amplitude:** (of the steady state solution) For oscillatory motion, $R = \sqrt{A^2 + B^2}$.

**Resonance:** Occurs when forcing frequency equals (or is very close to) the natural frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$.

1. (Sections 3.7, 3.8)
A string is stretched 15 cm by a force of 0.45 N. A mass of 0.3 kg is hung from the spring, and also attached to a viscous damper that exerts a force of 4 N when the velocity of the mass is 8 m/s. The mass is pulled down 6 cm below its equilibrium position and given an initial velocity of 12 cm/s downward.

   (a) Determine the position $u$ as a function of time $t$.

   (b) Find the quasifrequency of the motion.

   (c) If this system is also subjected to an external force $F(t) = 3\cos(5t)$, find $u(t)$, and the amplitude, period, and phase of the steady-state motion.

(a) $mu'' + cu' + ku = 0$ , $m = 0.3\text{kg}$, $c = \frac{4}{8} = 0.5\,\text{Ns/m}$

$u(0) = 0.06,\ u'(0) = 0.12$ $\quad k = \frac{0.45}{0.15} = 3\,\text{N/m}$

$0.3u'' + 0.5u' + 3u = 0 \Rightarrow 3u'' + 5u' + 30u = 0$

Solve: $3\lambda^2 + 5\lambda + 30 = 0 \Rightarrow \lambda = \frac{-5 \pm \sqrt{5^2 - 4\cdot 3\cdot 30}}{3\cdot 2} = \frac{-5 \pm \sqrt{335}\,i}{6}$

$$u(t) = c_1 e^{-5/6 t}\cos\left(\frac{\sqrt{335}}{6}t\right) + c_2 e^{-5/6 t}\sin\left(\frac{\sqrt{335}}{6}t\right)$$

Initial conds: $u(0) = c_1 = 0.06$, $u'(0) = -\frac{5}{6}\cdot 0.06 + \frac{\sqrt{335}}{6}c_2 = 0.12$

$$\Rightarrow c_2 = \frac{51}{50\sqrt{335}}$$

$$\boxed{u(t) = 0.06e^{-5/6 t}\cos\left(\frac{\sqrt{335}}{6}t\right) + \frac{51}{50\sqrt{335}}e^{-5/6 t}\sin\left(\frac{\sqrt{335}}{6}t\right) = u_h(t)}$$

(b) Quasi-frequency: $\boxed{\omega = \frac{\sqrt{335}}{6}\ \text{rad/sec}}$

(c) $u(t) = u_h(t) + u_p(t)$ where $u_p(t)$ is the steady-state solution

(c) $u_p(t) = A\cos(5t) + B\sin(5t)$ * method of undetermined coeffs *

$$u_p'(t) = -5A\sin(5t) + 5B\cos(5t)$$

$$u_p''(t) = -25A\cos(5t) - 25B\sin(5t)$$

$$\begin{aligned}3u_p'' + 5u_p' + 30u_p &= 3[-25A\cos(5t) - 25B\sin(5t)] \\ &\quad + 5[-5A\sin(5t) + 5B\cos(5t)] \\ &\quad + 30[A\cos(5t) + B\sin(5t)] \\ &= (-75A + 25B + 30A)\cos(5t) \\ &\quad + (-75B - 25A + 30B)\sin(5t) \\ &= 3\cos(5t)\end{aligned}$$

$$\Rightarrow -45A + 25B = 30, \quad -45B - 25A = 0 \Rightarrow A = -\frac{27}{53}, \quad B = \frac{15}{53}$$

$$u_p(t) = -\frac{27}{53}\cos(5t) + \frac{15}{53}\sin(5t)$$ (steady state motion)

$$\text{Amplitude} = \underbrace{\sqrt{\left(-\frac{27}{53}\right)^2 + \left(\frac{15}{53}\right)^2}}_{\sqrt{A^2+B^2}} = \frac{1}{53}\sqrt{27^2 + 15^2} = \boxed{\frac{\sqrt{954}}{53}}$$

$$\text{Period} = \frac{2\pi}{\omega} = \boxed{\frac{2\pi}{5} \text{ seconds}}$$ $\omega = 5$

Phase angle $\Rightarrow \tan\varphi = \dfrac{B}{A} = \dfrac{\frac{15}{53}}{-\frac{27}{53}} = -\dfrac{15}{27} \Rightarrow \varphi = \tan^{-1}\left(-\dfrac{15}{27}\right) + \pi$

(second quadrant)

## 2 6.1: Laplace Transforms

- The Laplace transform is defined by

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

- For many functions, you can just look up the Laplace transform in the table.

| $f(t)$ | $F(s)$ | defined for |
|---|---|---|
| $1$ | $\frac{1}{s}$ | $s > 0$ |
| $e^{at}$ | $\frac{1}{s-a}$ | $s > a$ |
| $t^n (n = 1, 2, \ldots)$ | $\frac{n!}{s^{n+1}}$ | $s > 0$ |
| $\sin(bt)$ | $\frac{b}{s^2 + b^2}$ | $s > 0$ |
| $\cos(bt)$ | $\frac{s}{s^2 + b^2}$ | $s > 0$ |
| $e^{at} t^n (n = 1, 2, \ldots)$ | $\frac{n!}{(s-a)^{n+1}}$ | $s > a$ |
| $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2 + b^2}$ | $s > a$ |
| $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2 + b^2}$ | $s > a$ |

- The Laplace transform is also linear:

$$\mathcal{L}\left\{c_1 f + c_2 g\right\} = c_1\mathcal{L}\{f\} + c_2\mathcal{L}\{g\}$$

- To take the inverse Laplace transform, you can also use the table. However, if your function does not match the things in the table, then you need to first do partial fractions.
- Partial fractions review
  - Simple roots
  - Irreducible quadratics
  - Repeated roots

2. (Section 6.1)
Find the Laplace transform of the following functions using the definition of Laplace transform

(a) $f(t) = \begin{cases} 2t, & 0 \le t < 1, \\ 3-t, & 1 \le t < 3, \\ 0, & t \ge 3 \end{cases}$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{f(t)\} = \int_0^1 e^{-st} \cdot 2t\,dt + \int_1^3 e^{-st}(3-t)\,dt + \int_3^\infty e^{-st} \cdot 0\,dt$

$= 2\int_0^1 t e^{-st}\,dt + 3\int_1^3 e^{-st}\,dt - \int_1^3 t e^{-st}\,dt$

$= -\frac{2t}{s} e^{-st}\Big|_0^1 - \frac{2}{s^2} e^{-st}\Big|_0^1 - \frac{3}{s} e^{-st}\Big|_1^3 + \frac{t}{s} e^{-st}\Big|_1^3 + \frac{1}{s^2} e^{-st}\Big|_1^3$

| $t$ | $e^{-st}$ |
|---|---|
| $1$ | $-\frac{1}{s} e^{-st}$ |
| $0$ | $\frac{1}{s^2} e^{-st}$ |

$= \cancel{-\frac{2}{s} e^{-s}} + 0 - \frac{2}{s^2} e^{-s} + \frac{2}{s^2} - \cancel{\frac{3}{s} e^{-3s}} + \cancel{\frac{3}{s} e^{-s}} + \cancel{\frac{3}{s} e^{-3s}} - \cancel{\frac{1}{s} e^{-s}} + \frac{1}{s^2} e^{-3s} - \frac{1}{s^2} e^{-s}$

$= \boxed{\frac{2}{s^2} - \frac{3}{s^2} e^{-s} + \frac{1}{s^2} e^{-3s}}$ * see page 10 *

(b) $g(t) = e^{-3t}\sin(2t)$.

$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} \cdot e^{-3t} \sin(2t)\,dt = \int_0^\infty e^{-(s+3)t} \sin(2t)\,dt$

| $\sin(2t)$ | $e^{-(s+3)t}$ |
|---|---|
| $2\cos(2t)$ | $\frac{-1}{s+3} e^{-(s+3)t}$ |
| $-4\sin(2t)$ | $\frac{1}{(s+3)^2} e^{-(s+3)t}$ |

$= -\frac{1}{s+3} e^{-(s+3)t} \sin(2t)\Big|_0^\infty - \frac{2}{(s+3)^2} e^{-(s+3)t} \cos(2t)\Big|_0^\infty$

$- \frac{4}{(s+3)^2} \int_0^\infty e^{-(s+3)t} \sin(2t)\,dt$

$G(s) + \frac{4}{(s+3)^2} G(s) = \frac{2}{(s+3)^2} \Rightarrow G(s)\left(1 + \frac{4}{(s+3)^2}\right) = \frac{2}{(s+3)^2}$

$\Rightarrow G(s) \frac{(s+3)^2+4}{(s+3)^2} = \frac{2}{(s+3)^2} \Rightarrow \boxed{G(s) = \frac{2}{(s+3)^2+4}}$

3. (Section 6.1) Use the table to find the Laplace transform of

$$f(t) = 4t^3 - 5\cos(\pi t) + 2t\sin(3t)$$

$$\mathcal{L}\{4t^3\} = 4 \cdot \frac{3!}{s^4} = \frac{24}{s^4}$$

$$\mathcal{L}\{-5\cos(\pi t)\} = -5 \cdot \frac{s}{s^2+\pi^2}$$

$$\mathcal{L}\{2t\sin(3t)\} = -2\frac{d}{ds}\mathcal{L}\{\sin(3t)\}$$ since $\mathcal{L}\{tf(t)\} = -F'(s)$

$$= -2\frac{d}{ds}\frac{3}{s^2+9}$$

$$= \frac{12s}{(s^2+9)^2}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{24}{s^4} - \frac{5s}{s^2+\pi^2} + \frac{12s}{(s^2+9)^2}}$$

4. (Sections 6.2, 6.3)
   Find the inverse Laplace transform of the function

$$F(s) = \frac{s+3}{(s^2+2s+5)(s-1)}$$

$$\frac{s+3}{(s^2+2s+5)(s-1)} = \frac{s+3}{[(s+1)^2+4](s-1)} = \frac{A(s+1)+B}{[(s+1)^2+4]} + \frac{C}{s-1}$$

$$s+3 = (s-1)[A(s+1)+B] + C[(s+1)^2+4]$$

$s=1$: $\quad 4 = 8C \Rightarrow C = 1/2$

$s=-1$: $\quad 2 = -2B + 4C = -2B + \frac{4}{2} = -2B + 2$

$\Rightarrow B = 0$

$s=0$: $\quad 3 = -A + 5C \Rightarrow A = 5C - 3 = \frac{5}{2} - 3 = -1/2$

$$F(s) = \frac{-1/2(s+1)}{[(s+1)^2+4]} + \frac{1/2}{s-1}$$

$$\mathcal{L}^{-1}\left\{\frac{-1/2(s+1)}{(s+1)^2+4}\right\} = -\frac{1}{2}e^{-t}\cos(2t) \qquad \mathcal{L}^{-1}\left\{\frac{1/2}{s-1}\right\} = \frac{1}{2}e^{t}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{2}e^{-t}\cos(2t) + \frac{1}{2}e^{t}}$$

# 3 6.2: Solving ODEs with Laplace Transforms

## Review

- Laplace transform of derivatives

$$\mathcal{L}\{f'\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

- How to solve differential equations with the Laplace transform
  - Apply the Laplace transform to each term of the ODE.
  - Substitute initial conditions and solve for $Y(s)$.
  - Compute the inverse Laplace transform to find $y(t)$.

5. (Section 6.2)
   Find the solution of the initial value problem

$$y'' + 4y' + 5y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1$$

using the method of Laplace transforms.

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$s^2 Y - s\cancel{y(0)} - y'(0) + 4[sY - \cancel{y(0)}] + 5Y = \frac{1}{s+1}$$

$$(s^2 + 4s + 5)Y - 1 = \frac{1}{s+1}$$

$$(s^2+4s+5)Y = 1 + \frac{1}{s+1} = \frac{s+2}{s+1}$$

$$Y = \frac{s+2}{(s^2+4s+5)(s+1)}$$

$$= \frac{s+2}{[(s+2)^2+1](s+1)} = \frac{A(s+2)+B}{(s+2)^2+1} + \frac{C}{s+1}$$

$$s+2 = (s+1)[A(s+2)+B] + C[(s+2)^2+1]$$

$s=-1$: $1 = 2C \Rightarrow C = 1/2$

$s=-2$: $0 = -B + C \Rightarrow B = C = 1/2$

$s=0$: $2 = 2A + B + 5C$

$2 - 1/2 - 5/2 = 2A$

$A = -1/2$

$$\boxed{y(t) = -\frac{1}{2}e^{-2t}\cos(t) + \frac{1}{2}e^{-2t}\sin(t) + \frac{1}{2}e^{-t}}$$

# 4 6.3: Step Functions

## Review

- The unit step function $u_c(t)$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

- It can be used to write discontinuous functions into a single equation.
- The Laplace transform of $u_c(t)$ is

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \quad s > 0$$

- Laplace transforms of shifts

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$
$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$$

- Inverse Laplace transform of shifts

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$$

(a) (Section 6.3)
Write the function below in terms of Heaviside unit step functions

$$f(t) = \begin{cases} 2t, & 0 \le t < 1, \\ 3 - t, & 1 \le t < 3, \\ 0, & t \ge 3 \end{cases}$$

$u_0 = 1$

$$f(t) = 2t(u_0 - u_1) + (3-t)(u_1 - u_3)$$

$$= 2t - 2tu_1 + 3u_1 - tu_1 - 3u_3 + tu_3$$

$$= \boxed{2t - 3u_1(t-1) + u_3(t-3)}$$

this form is convenient for Laplace transforms

(b) Find the Laplace transform of the function obtained in part (a).

use shift theorem

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2}{s^2} - \frac{3e^{-s}}{s^2} + \frac{e^{-3s}}{s^2}}$$

* see page 4 *

6. (Sections 6.2, 6.3)
   Find the inverse Laplace transform of the function

$$F(s) = \frac{e^{-3s}(s^2+2s+2)}{s(s+1)^2} = e^{-3s} \cdot \frac{s^2+2s+2}{s(s+1)^2}$$

$$\frac{s^2+2s+2}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$s^2+2s+2 = A(s+1)^2 + Bs(s+1) + Cs$$

$s=0$: $\quad 2 = A$

$s=-1$: $\quad 1 = -C \Rightarrow C = -1$

$s=1$: $\quad 5 = 4A + 2B + C$
$\quad = 8 + 2B - 1$
$\quad -2 = 2B \Rightarrow B = -1$

$$\frac{s^2+2s+2}{s(s+1)^2} = \frac{2}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+2s+2}{s(s+1)^2}\right\} = 2 - e^{-t} - te^{-t}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = u_3(t)\left[2 - e^{-(t-3)} - (t-3)e^{-(t-3)}\right]}$$

7. (Section 6.4)
Find the solution of the initial value problem using Laplace transforms

$$y'' + 2y' + 2y = \begin{cases} t, & 0 \le t < 3, \\ 3, & t \ge 3 \end{cases}, \quad y(0) = 1, \quad y'(0) = 0.$$

$\hookrightarrow t(u_0 - u_3) + 3u_3$

$= tu_0 - u_3(t-3)$

$u_0 = 1$

$y'' + 2y' + 2y = t - u_3(t-3)$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$s^2 Y - sy(0) - y'(0) + 2[sY - y(0)] + 2Y = \frac{1}{s^2} - \frac{e^{-3s}}{s^2}$

$(s^2 + 2s + 2)Y - s - 2 = \frac{1}{s^2}(1 - e^{-3s})$

$Y(s) = \frac{1 - e^{-3s}}{s^2(s^2+2s+2)} + \frac{s+2}{s^2+2s+2}$

$-\frac{1}{2} + \frac{t}{2} + \frac{1}{2}e^{-t}\cos(t)$

$\frac{1}{s^2(s^2+2s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s+1)+D}{(s+1)^2+1} = \boxed{\frac{-1}{2s} + \frac{1}{2s^2} + \frac{\frac{1}{2}s + \frac{1}{2}}{(s+1)^2+1}}$

$\frac{2+s}{s^2+2s+2} = \frac{s+1+1}{(s+1)^2+1} = \boxed{\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}} \rightarrow e^{-t}\cos(t) + e^{-t}\sin(t)$

$$\boxed{y(t) = \left[-\frac{1}{2} + \frac{t}{2} + \frac{1}{2}e^{-t}\cos(t)\right] - u_3(t)\left[-\frac{1}{2} + \frac{(t-3)}{3} + \frac{1}{2}e^{-(t-3)}\cos(t-3)\right] + e^{-t}\cos(t) + e^{-t}\sin(t)}$$

# 5 6.5: Delta Functions

## Review

- The Dirac delta function $\delta(t-c)$ is defined by

$$\int_{-\infty}^{\infty} f(t)\delta(t-c)\,dt = f(c), \quad c \geq 0$$

- It models instantaneous impulses.
- The Laplace transform of $\delta(t-c)$ is

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}$$

- Laplace transforms involving delta functions

$$\mathcal{L}\{g(t)\delta(t-c)\} = g(c)e^{-cs}$$

8. (Section 6.5)
Find the Laplace transform of the following function:

$$f(t) = t^4\delta(t-4) + e^{3t}\delta(t-2) + \sin(t)\cdot\delta\left(t - \frac{\pi}{2}\right)$$

use $\mathcal{L}\{f(t)\delta(t-c)\} = f(c)e^{-cs}$

$\mathcal{L}\{t^4\delta(t-4)\} = 4^4 e^{-4s}$

$\mathcal{L}\{e^{3t}\delta(t-2)\} = e^6 e^{-2s}$

$\mathcal{L}\{\sin(t)\cdot\delta(t-\frac{\pi}{2})\} = \sin(\frac{\pi}{2}) e^{-\frac{\pi}{2}s} = e^{-\frac{\pi}{2}s}$

$$\boxed{\mathcal{L}\{f(t)\} = 256e^{-4s} + e^6\cdot e^{-2s} + e^{-\frac{\pi}{2}s}}$$

9. (Section 6.5)
A 2 kg mass is suspended from a spring and damper. When the mass is hung at rest, it stretches the spring by 2 meters. When the mass moves at 1 m/s, the damper exerts a resistive force of 4 N. At $t = 2$ seconds, the system is struck with a hammer, delivering an instantaneous impulse force of magnitude 3 N. The mass starts motion from equilibrium with an initial upward velocity of 0.5 m/s.

   (a) Determine the spring constant $k$ and damping coefficient $c$.
   (b) Write the governing differential equation for the displacement $u(t)$.
   (c) Solve for $u(t)$ and describe the motion of the system.

(Use $g = 10\,\mathrm{m/s^2}$.)

(a) $k = \dfrac{mg}{u} = \dfrac{2 \cdot 10}{2} = 10\ \mathrm{N/m}$, $c = \dfrac{4\ \mathrm{N}}{1\ \mathrm{m/s}} = 4\ \mathrm{N \cdot s/m}$

(b) $2u'' + 4u' + 10u = \underbrace{3\delta(t-2)}_{\text{impulse at } t=2}$, $u(0) = 0$, $u'(0) = -0.5$

(c) Use Laplace transforms

$$2s^2 U - 2\cancel{u(0)} - 2u'(0) + 4sU - 4\cancel{u(0)} + 10U = 3e^{-2s}$$

$$(2s^2 + 4s + 10)U = 1 + 3e^{-2s} \Rightarrow U = \frac{1 + 3e^{-2s}}{2s^2 + 4s + 10}$$

$$\frac{1}{2s^2+4s+10} = \frac{1}{2(s^2+2s+5)} = \frac{1}{2[(s+1)^2+4]} = \frac{2}{4[(s+1)^2+4]}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2s^2+4s+10}\right\} = \frac{1}{4}e^{-t}\sin(2t)$$

$$\boxed{u(t) = \underbrace{\frac{1}{4}e^{-t}\sin(2t)}_{\substack{\text{natural response} \\ \text{(homogeneous solution)}}} + \underbrace{\frac{3}{4}u_2(t)\left[e^{-(t-2)}\sin[2(t-2)]\right]}_{\substack{\text{impulse response} \\ \text{(particular solution)}}}}$$

# 6 6.6: Convolution

## Review

- The convolution of two functions $f(t)$ and $g(t)$, denoted $(f * g)(t)$, is defined by

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx$$

  for $t \geq 0$, assuming both functions are zero for $t < 0$.

- The Laplace transform of a convolution $(f * g)(t)$ is

$$\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s),$$

  where $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$.

- This property simplifies solving differential equations by converting convolution in the time domain to multiplication in the s-domain.

- Convolution is commutative: $f * g = g * f$ so

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t f(t-x)g(x)\,dx.$$

10. (a) (Section 6.6)
Suppose $f(t) = e^{-t}$ and $g(t) = t^2$, find $(f * g)(t)$ using the definition of convolution.

$$(f*g)(t) = \int_0^t f(t-x)\,g(x)\,dx$$
$$= \int_0^t e^{-(t-x)} x^2\,dx = e^{-t}\int_0^t e^{x} \cdot x^2\,dx$$

| | |
|---|---|
| $x^2$ (+) | $e^x$ |
| $2x$ (−) | $e^x$ |
| $2$ (+) | $e^x$ |
| $0$ | $e^x$ |

$$= e^{-t}\left[x^2 e^{x} - 2x e^{x} + 2e^{x}\right]_0^t$$
$$= e^{-t}\left[t^2 e^{t} - 2t e^{t} + 2e^{t} - 2\right]$$
$$= \boxed{t^2 - 2t + 2 - 2e^{-t}}$$

(b) Use part (a) to determine $\mathcal{L}^{-1}\{H(s)\}$ where

Convolution theorem:

$$H(s) = \frac{2}{s^3(s+1)} = \left(\frac{2}{s^3}\right)\left(\frac{1}{s+1}\right)$$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = t^2 * e^{-t}$$

From 10(a) $\mathcal{L}^{-1}\left\{\frac{2}{s^3(s+1)}\right\} = \boxed{t^2 - 2t + 2 - 2e^{-t}}$

Comment: can get the same result from partial fractions

$$\frac{2}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} \quad \text{etc ....}$$

11. (Section 6.6)
Use convolution to determine the solution of the integro-differential equation

$$y' + 3\int_0^t y(t-x)\cos(x)\,dx = 2\sin(t), \quad y(0) = 0.$$

$$y' + 3\,y(t) * \cos(t) = 2\sin(t)$$

Take Laplace Transforms, and use the Convolution Theorem

$$sY - \cancel{y(0)} + 3Y \cdot \frac{s}{s^2+1} = \frac{2}{s^2+1}$$

$$\left(s + \frac{3s}{s^2+1}\right)Y = \frac{2}{s^2+1}$$

$$\left(\frac{s^3+4s}{s^2+1}\right)Y = \frac{2}{s^2+1} \Rightarrow Y = \frac{2\cancel{(s^2+1)}}{(s^3+4s)\cancel{(s^2+1)}}$$

$$= \frac{2}{s(s^2+4)}$$

$$\frac{2}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \Rightarrow 2 = A(s^2+4) + s[Bs+C]$$

$s=0$: $2 = 4A \Rightarrow A = 1/2$

$0 = A + B \Rightarrow B = -1/2$ ($s^2$ terms), $C = 0$ ($s$ terms)

$$Y = \frac{1/2}{s} - \frac{\frac{1}{2}s}{s^2+4} \Rightarrow y(t) = \frac{1}{2} - \frac{1}{2}\cos(2t) = \boxed{\frac{1}{2}(1-\cos(2t))}$$

# 7 5.1–5.2: Power Series Solutions of Linear Differential Equations

## Review

- A power series solution of a linear differential equation assumes the solution can be written as

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $x_0$ is the center of the series (often $x_0 = 0$), and $a_n$ are coefficients to be determined.

- The derivatives of the power series are:

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

obtained by term-by-term differentiation, assuming the series converges in some interval.

- For a linear differential equation of the form $P(x)y'' + Q(x)y' + R(x)y = 0$, substitute the power series for $y$, $y'$, and $y''$ into the equation, equate coefficients of like powers of $(x - x_0)$, and solve for the recurrence relation among the $a_n$.

- A point $x_0$ is an *ordinary point* if $P(x_0) \neq 0$ and the coefficients $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic at $x_0$. In this case, the series solution converges in some interval around $x_0$.

- The radius of convergence of the series solution is at least as large as the distance from $x_0$ to the nearest singular point (where $P(x) = 0$), determined by analyzing the coefficient functions.

- Solutions typically yield two linearly independent series $y_1(x)$ and $y_2(x)$, whose Wronskian $W[y_1, y_2](x_0) \neq 0$ confirms their independence.

12. (Section 5.1) Find the radius and interval of convergence of the series

(a)

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x+4)^n}{(n+1)!}$$

ratio test:

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\,(x+4)^{\cancel{n+1}}}{(n+2)!}\cdot\frac{\cancel{(n+1)!}}{\cancel{(-1)^n}\,\cancel{(x+4)^n}}\right| = \lim_{n\to\infty}\left|\frac{x+4}{n+2}\right|$$

$$= |x+4|\lim_{n\to\infty}\frac{1}{n+2} = 0 < 1$$

The limit is $0<1$ for every $x \Rightarrow \boxed{R=\infty}$

i.e. the series converges for all real numbers.
interval of convergence $-\infty < x < \infty$

(b)

$$\sum_{n=0}^{\infty}\frac{(x-1)^n}{2^n}$$

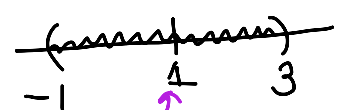
ratio test:

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x-1)^n}\right| = \lim_{n\to\infty}\frac{|x-1|}{2} < 1$$

$$\Rightarrow \frac{|x-1|}{2} < 1$$

$$\Rightarrow |x-1| < 2$$ radius of convergence

$R=2$




centered @ $x_0=1$

interval of convergence: $-1 < x < 3$

13. (Section 5.2) Consider the initial value value problem

$$y'' + x^2y' + 2xy = 0, \;\; y(0) = 1, \;\; y'(0) = 0.$$

(a) Solve the initial value problem using a series of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find the recurrence relation.

(b) Find the first 6 terms of the series solution.

(c) Write down the solution using summation notation.

(a) $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \Rightarrow x^2 y' = \sum_{n=0}^{\infty} n a_n x^{n+1} \underset{\text{(shift)}}{=} \sum_{n=1}^{\infty} (n-1) a_{n-1} x^n$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \underset{\text{(shift)}}{=} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$2xy = 2x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2a_n x^{n+1} \underset{\text{(shift)}}{=} \sum_{n=1}^{\infty} 2a_{n-1} x^n$

$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n$

adding the terms

$y'' + x^2 y' + 2xy = 2a_2 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) a_{n+2} + (n-1) a_{n-1} + 2a_{n-1} \right] x^n = 0$

$\Rightarrow 2a_2 = 0$, $(n+2)\cancel{(n+1)} a_{n+2} + \cancel{(n+1)} a_{n-1} = 0 \Rightarrow \boxed{a_{n+2} = \dfrac{-a_{n-1}}{n+2}, \; n \geq 1}$

$\hookrightarrow a_2 = 0$

recurrence relation

(b) $a_0 = y(0) = 1$, $a_1 = y'(0) = 0$, $a_2 = 0$

$n=1$: $a_3 = -\dfrac{a_0}{3} = -\dfrac{1}{3}$, $a_4 = \dfrac{-a_1}{4} = 0$, $a_5 = \dfrac{-a_2}{5} = 0$, $a_6 = -\dfrac{a_3}{6} = \dfrac{1}{6.3}$

$$\boxed{y(x) = 1 - \frac{1}{3}x^3 + \frac{1}{18}x^6 + \dots}$$

(c) Only terms of the form $x^{3n}$, $n=0,1,2,3,\ldots$ are non-zero.

$$y(x) = \sum_{n=0}^{\infty} a_{3n} x^{3n}$$

General term: $a_{3n} = (-1)^n \cdot \dfrac{1}{(3n)\cdots 9\cdot 6\cdot 3}$

multiply all multiples of 3 up to 3n

$$a_{3n} = (-1)^n \frac{1}{3^n \cdot n!}$$

factor a 3 from each of the n-terms

$$\Rightarrow \boxed{y(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{3^n n!} x^{3n}}$$

Note that: $y(x) = \displaystyle\sum_{n=0}^{\infty} \frac{\left(-\frac{x^3}{3}\right)^n}{n!} = e^{-x^3/3}$

since $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ & replace $x$ by $-x^3/3$