FINAL EXAM REVIEW

Exercise 1

Find the solution. Where is the solution defined?

$$y' - ty^2 = t, \qquad y(0) = 1.$$

$$\frac{dy}{dt} = t(1+y^2)$$

$$\int \frac{dy}{1+y^2} = \int t\,dt$$

$$\arctan(y) = \frac{1}{2}t^2 + c$$

$$y(t) = \tan\left(\frac{1}{2}t^2 + c\right)$$

$$y(0) = \tan(c) = 1$$

$$c = \pi/4$$

$$\boxed{y(t) = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{4}\right)}$$

Exercise 2

Find the general solution.

$$(\sin(x) + x^2 e^y)y' + y\cos(x) + 2xe^y - 2 = 0$$

$$\underbrace{\left(y\cos(x) + 2xe^y - 2\right)}_{M} + \underbrace{\left(\sin(x) + x^2 e^y\right)}_{N} y' = 0$$

$$M_y = \cos(x) + 2xe^y = N_x = \cos(x) + 2xe^y \quad \checkmark \text{ exact}$$

$$\Psi_x = y\cos(x) + 2xe^y - 2 \qquad \Psi = y\sin(x) + x^2 e^y - 2x + C(y)$$

$$\Rightarrow$$

$$\Psi_y = \sin(x) + x^2 e^y \qquad \Psi = y\sin(x) + x^2 e^y + C(x)$$

$$\Psi(x,y) = \boxed{y\sin(x) + x^2 e^y - 2x = C}$$

Exercise 3

Without solving the equation, determine where a unique solution is guaranteed to exist.

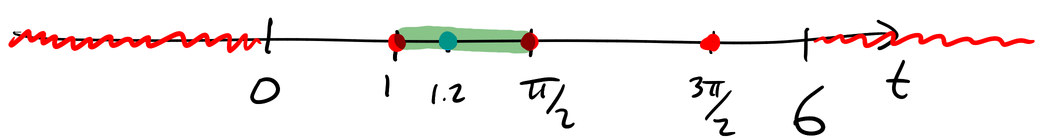
$$\ln(t)y'' - y' + \tan(t)y = \sqrt{6-t}, \qquad y(6/5) = 16/5.$$

$t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

$6 - t \geq 0$

$6 \geq t$

$$y'' - \underbrace{\frac{1}{\ln(t)}}_{t>0,\ t\neq 1} y' + \frac{\tan(t)}{\ln(t)} y = \frac{\sqrt{6-t}}{\ln(t)}$$



There exists a unique solution on the interval $(1, \pi/2)$.

Exercise 4

Consider the differential equation

$$y' = y^3 - 5y^2 + 6y.$$

(a) Find the equilibrium solutions.

(b) Plot the direction field.

- Draw a few example solutions on the direction field.

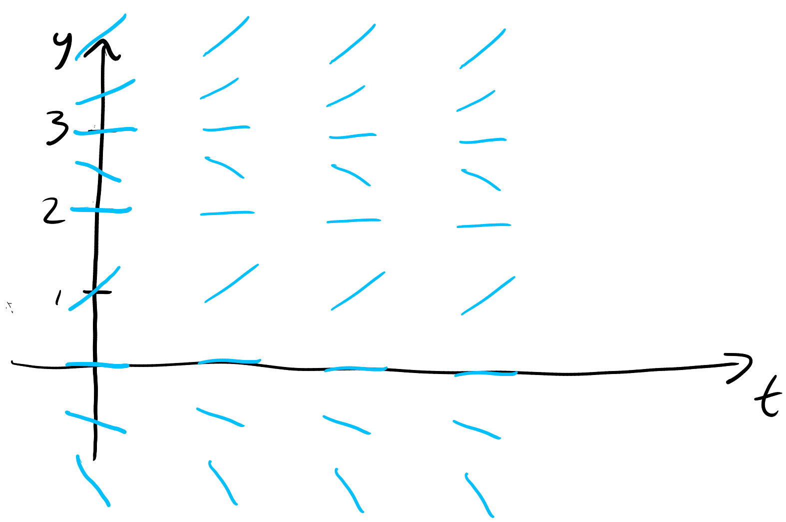
(c) Draw the phase line diagram.

(d) Determine the stability of each equilibrium point.

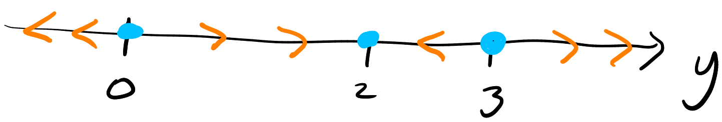
(a) $y' = 0 = y(y^2 - 5y + 6) = y(y-2)(y-3)$

$$\boxed{y = 0, 2, 3}$$

(b)



(c)



(d) 0 and 3 are unstable

2 is (asymptotically) stable

Exercise 5

Suppose you take out a \$20,000 loan with a 7% annual interest rate that is compounded continuously. Suppose you pay \$300 per month on the loan. Write down a differential equation to describe the amount of the loan. How long will it take you to pay off the loan?

$L(t)$ = loan amount (in \$)

t = time (in years)

$$\boxed{\frac{dL}{dt} = 0.07L - 3600, \quad L(0) = \$20{,}000}$$

$$\mu L' \underbrace{- 0.07\mu L}_{\frac{d\mu}{dt}} = -3600\mu$$

$$\frac{d\mu}{dt} = -0.07\mu \Rightarrow \mu(t) = e^{-0.07t}$$

$$\frac{d}{dt}\left(e^{-0.07t}L(t)\right) = -3600e^{-0.07t}$$

$$e^{-0.07t}L(t) = \frac{3600}{0.07}e^{-0.07t} + C$$

$$L(t) = \frac{3600}{0.07} + Ce^{0.07t}$$

$$L(0) = \frac{3600}{0.07} + c = 20{,}000$$

$$\Rightarrow c = 20{,}000 - \frac{3600}{0.07}$$

$$L(t) = \frac{3600}{0.07} + \left(20{,}000 - \frac{3600}{0.07}\right)e^{0.07t}$$

Loan is paid off when $L(t) = 0$:

$$\frac{3600}{0.07} + \left(20{,}000 - \frac{3600}{0.07}\right)e^{0.07t} = 0$$

$$e^{0.07t} = \frac{-\frac{3600}{0.07}}{20{,}000 - \frac{3600}{0.07}}$$

$$\boxed{t = \frac{1}{0.07} \ln\left(\frac{-\frac{3600}{0.07}}{20{,}000 - \frac{3600}{0.07}}\right) \text{ years}}$$

Exercise 6

Find an integrating factor that makes the following equation exact.

$$\underbrace{(3xy+y^2)}_{M}+\underbrace{(x^2+xy)}_{N}y'=0$$

μ only depends on x:

$$\frac{d\mu}{dx}=\mu\cdot\left(\frac{M_y-N_x}{N}\right)$$

$$=\mu\cdot\left(\frac{3x+2y-(2x+y)}{x^2+xy}\right)$$

$$=\mu\cdot\left(\frac{3x+2y-2x-y}{x(x+y)}\right)$$

$$=\mu\cdot\left(\frac{x+y}{x(x+y)}\right)$$

$$=\mu\cdot\underbrace{\left(\frac{1}{x}\right)}_{\text{doesn't depend on } y\ \checkmark}$$

$$\int\frac{d\mu}{\mu}=\int\frac{dx}{x}$$

$$\ln|\mu|=\ln|x|$$

$$\boxed{\mu(x)=x}$$

μ only depends on y:

$$\frac{d\mu}{dy}=\mu\cdot\left(\frac{N_x-M_y}{M}\right)$$

Exercise 7

Find the general solution.

$$u'' + 4u' + 4u = 0$$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$

$r = -2$

$$\boxed{u(t) = c_1 e^{-2t} + c_2 t e^{-2t}}$$

Exercise 8

Find the general solution. Prove that it is indeed the general solution.

$$u'' + 4u' + 5u = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(5)}}{2} = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm \frac{2i}{2} = -2 \pm i$$

$$u(t) = c_1 \underbrace{e^{-2t}\cos(t)}_{u_1} + c_2 \underbrace{e^{-2t}\sin(t)}_{u_2}$$

$$u_1'(t) = -2e^{-2t}\cos(t) - e^{-2t}\sin(t)$$

$$u_2'(t) = -2e^{-2t}\sin(t) + e^{-2t}\cos(t)$$

$$W[u_1, u_2](0) = \begin{vmatrix} u_1(0) & u_2(0) \\ u_1'(0) & u_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1 \neq 0$$

So, u_1 and u_2 form a fundamental set of solutions. i.e., $c_1 u_1 + c_2 u_2$ is the general solution.

Exercise 9

Find a particular solution to

$$t^2y'' - 2y = 3t^2, \qquad t > 0,$$

given that $\underbrace{t^2}_{y_1}$ and $\underbrace{t^{-1}}_{y_2}$ are solutions to the corresponding homogeneous equation.

$$y'' - 2t^{-2}y = \underbrace{3}_{g(t)}$$

$$W[t^2, t^{-1}](t) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3$$

$$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}\,dt$$

$$= -t^2\int \frac{t^{-1}\cdot 3}{-3}\,dt + t^{-1}\int \frac{t^2\cdot 3}{-3}\,dt$$

$$= t^2\int t^{-1}\,dt - t^{-1}\int t^2\,dt$$

$$= t^2\ln(t) - t^{-1}\cdot\frac{1}{3}t^3$$

(we just want one particular solution, so the +c doesn't matter)

$$\boxed{= t^2\ln(t) - \frac{1}{3}t^2}$$

Exercise 10

Find a particular solution.

$$y'' - 3y' - 10y = 2e^{5t} + 1$$

homogeneous solution:

$r^2 - 3r - 10 = 0$

$(r-5)(r+2) = 0$

$r = -2, 5$

$y_h(t) = c_1 e^{-2t} + c_2 e^{5t}$

particular solution:

$y_p(t) = Ate^{5t} + B$

$y_p'(t) = 5Ate^{5t} + Ae^{5t}$

$y_p''(t) = 25Ate^{5t} + 5Ae^{5t} + 5Ae^{5t} = 25Ate^{5t} + 10Ae^{5t}$

plug into diff eq:

$25Ate^{5t} + 10Ae^{5t} - 3(5Ate^{5t} + Ae^{5t}) - 10(Ate^{5t} + B)$
$= 2e^{5t} + 1$

$\cancel{25Ate^{5t}} + 10Ae^{5t} \cancel{- 15Ate^{5t}} - 3Ae^{5t} \cancel{- 10Ate^{5t}} - 10B$
$= 2e^{5t} + 1$

$$\underbrace{7A}_{=2} e^{5t} \underbrace{-10B}_{=1} = 2e^{5t} + 1$$

$$7A = 2 \qquad -10B = 1$$

$$A = 2/7 \qquad B = -1/10$$

$$\boxed{y_p(t) = \frac{2}{7} t e^{5t} - \frac{1}{10}}$$

Exercise 11

Suppose there is a 20 N mass hanging on a spring. When the mass was attached to the spring, the spring stretched by 30 cm. When the mass is moving 3 m/s, it experiences a damping force of 12 N. There is an external upward force of 7 N acting on the mass for the first 9 seconds, after which there is no external force. Initially the mass sent into motion with a downward velocity of 40 cm/s from the equilibrium position. (Use $g = 10\,\mathrm{m/s^2}$.)

(a) Write down an initial value problem that describes the motion of the mass.

$$m = \frac{20\,N}{10\,m/s^2} = 2\,kg$$

$$\gamma = \frac{12\,N}{3\,m/s} = 4\,\frac{N}{m/s}$$

$$k = \frac{mg}{L} = \frac{20N}{0.3m} = \frac{200}{3}\,\frac{N}{m}$$

$$\boxed{\begin{gathered} 2u'' + 4u' + \frac{200}{3}u = -7 + 7u_9(t) \\ u(0) = 0\,m, \quad u'(0) = 0.4\,m/s \end{gathered}}$$

(b) Is the system $\underbrace{\text{over}}_{}$, $\underbrace{\text{under}}_{}$, or $\underbrace{\text{critically damped}}_{}$?

distinct complex repeated

$$2r^2 + 4r + \frac{200}{3} = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(2)(\frac{200}{3})}}{2(2)} = -1 \pm \frac{\sqrt{16 - \frac{1600}{3}}}{4}$$

negative under squareroot, so complex roots.

$\Rightarrow \boxed{\text{underdamped}}$

Exercise 12

Solve the initial value problem.

typo fixed

$$y'' - 3y' + 2y = u_1(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

$$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} - 3(sY(s) - \cancel{y(0)}^{0}) + 2Y(s) = \frac{e^{-s}}{s}$$

$$(s^2 - 3s + 2)Y(s) = \frac{e^{-s}}{s}$$

$$Y(s) = e^{-s}\left(\frac{1}{s(s^2-3s+2)}\right) = e^{-s}\underbrace{\left(\frac{1}{s(s-1)(s-2)}\right)}_{= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}}$$

$$1 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

$s=0$: $1 = 2A \Rightarrow A = 1/2$

$s=1$: $1 = -B \Rightarrow B = -1$

$s=2$: $1 = 2C \Rightarrow C = 1/2$

$$Y(s) = e^{-s}\left(\frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2}\right)$$

$$\boxed{y(t) = u_1(t)\left(\frac{1}{2} - e^{t-1} + \frac{1}{2}e^{2(t-1)}\right)}$$

Exercise 13

Using the definition of the Laplace transform, show that $\mathcal{L}\{t\} = \frac{1}{s^2}$.

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$u = t$ $\quad dv = e^{-st}dt$

$du = dt$ $\quad v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s} t e^{-st}\Big|_0^\infty + \int_0^\infty \frac{+1}{s} e^{-st} dt$$

$$= -\frac{1}{s}\Big(\underbrace{\lim_{t\to\infty} te^{-st}}_{0 \text{ if } s>0} - 0\Big) - \frac{1}{s^2} e^{-st}\Big|_0^\infty$$

$$= \frac{-1}{s^2}\Big(\underbrace{\lim_{t\to\infty} e^{-st}}_{0 \text{ if } s>0} - 1\Big)$$

$$\boxed{= \frac{1}{s^2} \quad \text{for} \quad s > 0}$$

Exercise 14

Find the general solution in the form of a power series centered at $x = 1$.

$$y'' - xy = 0.$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$- (x-1+1) \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$- (x-1) \sum_{n=0}^{\infty} a_n (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$- \sum_{n=0}^{\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$n \to n+2$ $\qquad$ $n \to n-1$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - a_0 - \sum_{n=1}^{\infty} a_n (x-1)^n = 0$$

$$\underbrace{2a_2 - a_0}_{=0} + \sum_{n=1}^{\infty} \underbrace{\left[(n+2)(n+1) a_{n+2} - a_{n-1} - a_n \right]}_{=0} = 0$$

$$a_2 = \frac{1}{2} a_0 \qquad a_{n+2} = \frac{a_{n-1} + a_n}{(n+2)(n+1)} \qquad \text{for } n = 1, 2, 3, \ldots$$

$$a_3 = \frac{a_0 + a_1}{6} = \frac{1}{6} a_0 + \frac{1}{6} a_1$$

$$a_4 = \frac{a_1 + a_2}{12} = \frac{1}{12} a_1 + \frac{1}{12}\left(\frac{1}{2} a_0\right) = \frac{1}{24} a_0 + \frac{1}{12} a_1$$

$$\vdots$$

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$= a_0 + a_1(x-1) + \frac{1}{2} a_0 (x-1)^2 + \left(\frac{1}{6} a_0 + \frac{1}{6} a_1\right)(x-1)^3$$

$$+ \left(\frac{1}{24} a_0 + \frac{1}{12} a_1\right)(x-1)^4 + \cdots$$

$$= a_0 \underbrace{\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \cdots\right)}_{y_1(x)}$$

$$+ a_1 \underbrace{\left((x-1) + \frac{1}{6} a_1 (x-1)^3 + \frac{1}{12}(x-1)^4 + \cdots\right)}_{y_2(x)}$$

General solution: $\boxed{y(x) = a_0 y_1(x) + a_1 y_2(x)}$

Exercise 15

Find the general solution to the system of differential equations.

$$x_1' = x_1 - 4x_2$$
$$x_2' = 4x_1 - 7x_2$$

$$\vec{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \vec{x}$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)^2 = 0$$

$$r = -3 \qquad \text{(repeated eigenvalue)}$$

eigenvector for $r = -3$:

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow 4\xi_1 - 4\xi_2 = 0 \Rightarrow \xi_1 = \xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_1 \end{bmatrix} = \cancelto{1}{\xi_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

generalized eigenvector:

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \vec{\eta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow 4\eta_1 - 4\eta_2 = 1 \Rightarrow \eta_1 = \frac{1}{4} + \eta_2$$

$$\vec{\eta} = \begin{bmatrix} 1/4 + \eta_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} + \cancelto{0}{\eta_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{x}(t) = c_1 e^{-3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2\left(te^{-3t}\begin{bmatrix}1\\1\end{bmatrix} + e^{-3t}\begin{bmatrix}1/4\\0\end{bmatrix}\right)}$$