# 24C_WIR_M140_H11_Kapita

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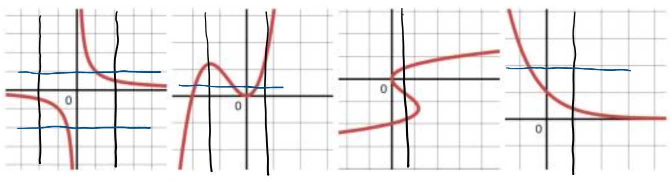
## Math 140: Week-in-Review 11 (Chapters 5.7, 5.8, Review over 5.1-5.6)

vertical line test (VLT) horizontal line test (HLT)

1. Which of the following graphs represent functions? Which functions are one-to-one?



| | | | |
|---|---|---|---|
| * passes both the VLT & the HLT | * passes VLT | * fails the VLT | * passes VLT |
| | * fails HLT | | * passes HLT |
| * function ✓ | * function ✓ | * function ✗ | * function ✓ |
| * One-to-one ✓ | * One-to-one ✗ | | * One-to-one ✓ |

2. Write the following expression as a single logarithm: $\frac{2}{3}(4\log_b(x) + \log_b(3x+1) - 5\log_b(2))$

$$\frac{2}{3}(4\log_b(x) + \log_b(3x+1) - 5\log_b(2)) = \frac{2}{3}\left(\log_b(x^4) + \log_b(3x+1) - \log_b(2^5)\right)$$

$$= \frac{2}{3}\left(\log_b\left(x^4(3x+1)\right) - \log_b(32)\right)$$

$$= \frac{2}{3}\log_b\left(\frac{x^4(3x+1)}{32}\right)$$

$$= \boxed{\log_b\left(\left(\frac{x^4(3x+1)}{32}\right)^{2/3}\right)}$$

3. Use properties of logarithms to fully expand the following:

$$\ln\left(\left(\sqrt[5]{\frac{4e\cdot x^2}{y^3\cdot z^{-4}}}\right)^8\right) = \ln\left(\left(\frac{4ex^2}{y^3 z^{-4}}\right)^{8/5}\right)$$

$$= \frac{8}{5}\ln\left(\frac{4ex^2}{y^3\cdot z^{-4}}\right)$$

$$= \frac{8}{5}\left[\ln(4ex^2) - \ln(y^3 z^{-4})\right]$$

$$= \frac{8}{5}\left[\ln(4) + \cancel{\ln(e)} + 2\ln(x) - 3\ln(y) - (-4)\ln(z)\right]$$

1

$$= \boxed{\frac{8}{5}\left[\ln(4) + 2\ln(x) - 3\ln(y) + 4\ln(z)\right]}$$

4. If $\log_b(3) = 0.428$ and $\log_b(5) = 0.627$ are approximations to 3 decimal places, find an approximation of $\log_b\left(\frac{27b}{125}\right)$.

$27 = 3^3$, $125 = 5^3$, $\log_b b = 1$

$$\log_b\left(\frac{27b}{125}\right) = \log_b(3^3 b) - \log_b(5^3) = 3\log_b(3) + \overbrace{\log_b(b)}^{1} - 3\log_b(5)$$

$$= 3\cdot(0.428) + 1 - 3\cdot(0.627)$$

$$= \boxed{0.403}$$

5. Write each logarithmic expression in its equivalent exponential form.

$y = \log_b(x) \Leftrightarrow b^y = x$

* remember these! *

(a) $\log_{121}(11) = \frac{1}{2}$

$\boxed{121^{1/2} = 11}$ OR $\left(\sqrt{121} = 11\right)$

(b) $\log_{1/6}(36) = -2$

$\boxed{\left(\frac{1}{6}\right)^{-2} = 36}$

check: $\left(\frac{1}{6}\right)^{-2} = \left(6^{-1}\right)^{-2} = 6^2 = 36$ ✓

(c) $x = \log_y(x+y)$

$\boxed{y^x = x + y}$

6. Write each exponential expression in its equivalent logarithmic form.

$b^y = x \Leftrightarrow y = \log_b(x)$

(a) $0.00001 = 10^{-5}$

base = 10

$\boxed{\log_{10}(0.00001) = -5}$

* check! $\log_{10}(0.00001) = \log_{10}(10^{-5})$ ✓

$= -5\underbrace{\log_{10}(10)}_{1} = -5$

(b) $\frac{8}{27} = \left(\frac{2}{3}\right)^3$

base $= \frac{2}{3}$

$\boxed{\log_{\frac{2}{3}}\left(\frac{8}{27}\right) = 3}$

(c) $y = 7^{3x+5}$

base = 7

$\boxed{\log_7(y) = 3x + 5}$

7. Simplify each of the following **without** using a calculator.

$27 = 3^3$, $9 = 3^2$

(a) $\log_3\left(\frac{27}{9^{-2}}\right)$

$$\log_3\left(\frac{27}{9^{-2}}\right) = \log_3(27) - \log_3(9^{-2}) = \log_3(3^3) - \log_3(3^{2\cdot(-2)})$$

$$= 3\cancel{\log_3(3)} - (-4)\cancel{\log_3(3)}$$

$$= 3 + 4 = \boxed{7}$$

$\boxed{\log_b(b) = 1 \quad b > 0 \quad b \neq 1}$

(b) $3^{2\log_3 4}$

$$3^{2\log_3(4)} = 3^{\log_3(4^2)} = 3^{\log_3(16)} = \boxed{16}$$

$b=3,\ x=16$

$\boxed{b^{\log_b(x)} = x, \quad x > 0}$

base e

$\boxed{e^{\ln(x)} = x}$

(c) $-17e^{-2\ln x^2}$

$$-17e^{-2\ln(x^2)} = -17e^{\ln((x^2)^{-2})} = -17e^{\ln(x^{-4})}$$

$$= -17x^{-4}$$

$$= \boxed{-\frac{17}{x^4}}$$

$* \ln(x) = \log_e(x)$

8. Algebraically solve the following for $x$. Give **exact** answers not calculator approximations.

(a) $\ln(x^2) - \ln(4) = 0$

domain : $x^2 > 0 \Rightarrow (-\infty, 0) \cup (0, \infty)$

$$\ln(x^2) - \ln(4) = 0$$

$$\ln\left(\frac{x^2}{4}\right) = 0$$

$\ln(1) = 0$

$$\Downarrow$$

$$\ln\left(\frac{x^2}{4}\right) = \ln(1)$$

$$\frac{x^2}{4} = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

both in domain

$$\boxed{x = \pm 2}$$

(b) $16^{x-1} = 2^{x^2}$

$16 = 2^4$

$2^{4(x-1)} = 2^{x^2}$ base $= 2$

$4x - 4 = x^2$

$\Rightarrow 0 = x^2 - 4x + 4$

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - (4)(1)(4)}}{2} = \dfrac{4 \pm \sqrt{0}}{2} \Rightarrow \boxed{x = 2}$

(c) $\underbrace{\log_5(x-4)}_{x>4} + \underbrace{\log_5(x)}_{x>0} = 1$

domain: $x > 4$ $(4, \infty)$

$\boxed{1 = \log_5(5)}$

$\log_5(x(x-4)) = \log_5(5)$

$x^2 - 4x = 5 \Rightarrow x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0 \Rightarrow x = 5, -1$

extraneous solution

$\boxed{x = 5}$

(d) $e^{5x-1} = e^{x^2+3}$ base $e$

$5x - 1 = x^2 + 3$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-4)(x-1) = 0$

$\boxed{x = 4, 1}$

$x < 10$ $\quad x > -2$ $\quad -2 \quad 10$

(e) $\log_3(10 - x) - \log_3(x + 2) = 1$

$\boxed{1 = \log_3(3)}$

$\log_3\left(\frac{10-x}{x+2}\right) = \log_3(3)$

domain: $(-2, 10)$

$\frac{10-x}{x+2} = 3 \Rightarrow 10 - x = 3(x+2)$

$10 - x = 3x + 6$

$4 = 4x \Rightarrow \boxed{x = 1}$ → in $(-2, 10)$ ✓

(f) $64^x = 4^{2x-3}$

$64 = 4^3$

$4^{3x} = 4^{2x-3}$ base 4

$3x = 2x - 3$

$\boxed{x = -3}$

(g) $e^{2x} - 4e^x - 5 = 0$

Let $e^x = y$. Then $e^{2x} = (e^x)^2 = y^2$

$y^2 - 4y - 5 = 0$

$(y-5)(y+1) = 0$

$y = 5, \; y = -1 \Rightarrow \boxed{e^x = 5}, \; e^x = -1$

solve for $x$

$e^x = 5$

$\ln(e^x) = \ln(5)$

$\boxed{x = \ln(5)}$

no solns! $e^x > 0$ for all $x$

9. How long will it take for your money to double in an account paying 3.5% annual interest compounded continuously, assuming no additional deposits, charges, or withdrawals?

Let $P$ = initial deposit. $A = Pe^{rt}$, $r = 0.035$, $A = 2P$, Find $t$

$2\cancel{P} = \cancel{P}e^{rt} \Rightarrow 2 = e^{rt}$

$\ln(2) = rt \Rightarrow t = \frac{\ln(2)}{r} = \frac{\ln(2)}{0.035} = \boxed{19.80 \text{ years}}$

* the doubling time is independent of $P$! *
↳ feature of exponential growth

10. **Algebraically** determine the domain of each of the following functions. Write your answers using interval notation.

(a) $f(x) = \log_3(4 - x)$

* logarithm: $4 - x > 0 \Rightarrow 4 > x$
$x < 4$

Domain of $f(x)$: $\boxed{(-\infty, 4)}$

(b) $g(x) = \dfrac{\sqrt{2x-5}}{\log_2(x^2+3)}$

* numerator: $\sqrt{2x-5}$ is an even root
$\Rightarrow 2x - 5 \geq 0 \Rightarrow \frac{2x}{2} \geq \frac{5}{2} \Rightarrow x \geq 5/2$

* denominator $\neq 0$, $\log_2(x^2+3)$ is defined
if $x^2 + 3 > 0 \Rightarrow x^2 > -3$ (always true)
& $\log_2(x^2+3) > 0$ (always true)
since $x^2 + 3 \geq 3 > 1$ ✓

Domain of $g(x)$: $\boxed{[5/2, \infty)}$

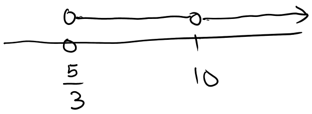
(c) $h(x) = \dfrac{\log_5(3x-5)}{\sqrt[7]{x-10}}$

* numerator defined if $3x - 5 > 0 \Rightarrow \frac{3x}{3} > \frac{5}{3}$

$x > \frac{5}{3}$

* denominator is an odd root function $\Rightarrow$ defined for all $\underbrace{x-10}_{\text{polynom.}}$ $\Rightarrow$ defined for all $x$

* denominator $\neq 0 \Rightarrow x \neq 10$




Domain of $h(x)$: $\boxed{\left(\frac{5}{3}, 10\right) \cup (10, \infty)}$

(d) $j(x) = \dfrac{(3x+5)e^{2x-1}}{4x^3 - 4x^2 - 24x}$

* numerator: product of a polynomial $3x+5$ and an exponential $e^{2x-1}$

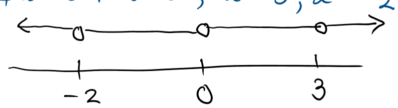
$\Rightarrow$ numerator is defined for all $x$

* denominator: polynomial $\Rightarrow$ defined for all $x$

* denominator $\neq 0$:

$$4x^3 - 4x^2 - 24x = 4x(x^2 - x - 6)$$
$$= 4x(x-3)(x+2)$$
$$= 0$$

$4x = 0 \Rightarrow x = 0$, $x = 3$, $x = -2$




Domain of $j(x)$:

$\boxed{(-\infty, -2) \cup (-2, 0) \cup (0, 3) \cup (3, \infty)}$

(e) $k(x) = \dfrac{(5x+11)^{1/4} + 3^{1/x}}{3e^x - 6} = \dfrac{\sqrt[4]{5x+11} + 3^{1/x}}{3e^x - 6}$

(even root)

$\ast$ numerator : $5x+11 \geq 0 \Rightarrow \frac{5x}{5} \geq \frac{-11}{5} \Rightarrow x \geq -\frac{11}{5}$ , $3^{1/x} \Rightarrow x \neq 0$

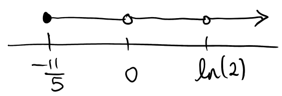
$1/x \rightarrow$ rational fn

$\hookrightarrow$ exponential

$\ast$ denominator : defined for every $x$ ✓

denom $\neq 0$ : $3e^x - 6 = 0 \Rightarrow \frac{3e^x}{3} = \frac{6}{3} \Rightarrow e^x = 2$

$e^x = 2 \Rightarrow \ln(e^x) = \ln(2) \Rightarrow x = \ln(2)$




Domain of $k(x)$: $\boxed{\left[-\frac{11}{5}, 0\right) \cup (0, \ln(2)) \cup (\ln(2), \infty)}$

(f) $\ell(x) = \begin{cases} \dfrac{e^x + 3}{\sqrt[4]{x+10}} & \text{if } -3 \leq x < 3, \\ \dfrac{x-4}{x^2 - 4x} & \text{if } 3 \leq x < 5, \\ 3x^9 + 5x^6 - 12x^4 + 103 & \text{if } x \geq 7. \end{cases}$

$\rightarrow x + 10 \geq 0 \Rightarrow x \geq -10$ ✗ always true in the domain $[-3, 3)$

$\rightarrow x^2 - 4x \neq 0 \Rightarrow x(x-4) \neq 0 \Rightarrow x \neq 0$ ✗, $x \neq 4$ ✓

$[3, 4) \cup (4, 5)$

$\rightarrow$ defined for $[7, \infty)$

$\boxed{\text{Domain of } \ell(x) : [-3, 4) \cup (4, 5) \cup [7, \infty)}$

11. Determine the zeros of $f(x) = \frac{2x(x+4)(3x-8)}{(x-3)(2x+5)}$

Domain of $f(x)$: $(x-3)(2x+5) \neq 0$

$x \neq 3$ & $x \neq \frac{-5}{2}$

$(-\infty, -\frac{5}{2}) \cup (\frac{-5}{2}, 3) \cup (3, \infty)$

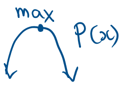
To find zeros, set numerator $= 0$

$2x(x+4)(3x-8) = 0$ (Zero product property)

$2x = 0$, $x+4 = 0$, $3x-8 = 0$

$\boxed{x = 0, \ x = -4, \ x = 8/3}$

12. A company selling gadgets has a price-demand function of $p(x) = -\frac{2}{225}x + 48$, where $x$ is the number of gadgets sold for a price of $p$ dollars. The company has a profit function of $P(x) = -\frac{2}{225}x^2 + 24x - 1800$.




(a) What is the maximum profit the company will earn?

Find the number of gadgets that maximize profit:

$$x = \frac{-b}{2a} = \frac{-24}{2\left(\frac{-2}{225}\right)} = \frac{12}{\left(\frac{2}{225}\right)} = \frac{12 * 225}{2} = 6 * 225 = 1350 \text{ gadgets}$$

$$P(1350) = -\frac{2}{225}(1350)^2 + 24(1350) - 1800 = 14\,400$$

Maximum profit $= \boxed{\$14,400}$

(b) What price will be charged in order to maximize profit?

* evaluate into the price function *

$$p(1350) = \frac{-2}{\cancel{225}_{1}}(\cancel{1350}^{6}) + 48 = -12 + 48 = 36$$

price of one gadget to maximize profit = \$36 per gadget

(c) Write the cost function for the company.

$$P(x) = R(x) - C(x) \Rightarrow C(x) = R(x) - P(x)$$

$$*R(x) = p(x)\cdot x = \left(\frac{-2}{225}x + 48\right)x = \frac{-2}{225}x^2 + 48x$$

$$* \quad C(x) = \left(\cancel{\frac{-2}{225}x^2} + 48x\right) - \left(\cancel{\frac{-2}{225}x^2} + 24x - 1800\right)$$

$$= \boxed{24x + 1800}$$

13. Compute and fully simplify the difference quotient of $f(x) = \frac{x}{5-x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\frac{x+h}{5-(x+h)} - \frac{x}{5-x}\right] = \frac{1}{h}\left[\frac{(x+h)(5-x) - x(5-x-h)}{(5-x-h)(5-x)}\right]$$

$$= \frac{1}{h}\left[\frac{\cancel{5x} - \cancel{x^2} + 5h - \cancel{xh} - \cancel{5x} + \cancel{x^2} + \cancel{xh}}{(5-x-h)(5-x)}\right]$$

$$= \frac{1}{\cancel{h}}\left[\frac{5\cancel{h}}{(5-x-h)(5-x)}\right]$$

$$= \boxed{\frac{5}{(5-x-h)(5-x)}}$$

14. Completely simplify the following expression, and write your answer using no negative exponents and no radicals

$$\frac{(5x^{-2}y^4)^{-2}z^8}{x^3\left(\sqrt[3]{x^2yz^{-3}}\right)}$$

$$\frac{5^{-2}\cdot x^{(-2)(-2)}\cdot y^{4(-2)}\cdot z^{8}}{x^3\left(x^2yz^{-3}\right)^{1/3}} = \frac{5^{-2}\cdot x^{4}\cdot y^{-8}\cdot z^{8}}{x^3\, x^{2/3}\cdot y^{1/3}\cdot z^{-1}}$$

$$= 5^{-2}\cdot x^{(4-3-2/3)}\cdot y^{-8-1/3}\cdot z^{8-(-1)}$$

$$= 5^{-2}\cdot x^{1/3}\cdot y^{-25/3}\cdot z^{9}$$

$$= \frac{x^{1/3}\cdot z^9}{5^2\cdot y^{25/3}}$$

$$= \boxed{\frac{x^{1/3}\cdot z^9}{25\, y^{25/3}}}$$

15. Write $f(x) = \left|15 - \frac{5}{2}x\right|$ as an equivalent piecewise-defined function.

$$\left|15 - \frac{5}{2}x\right| = \begin{cases} -(15 - \frac{5}{2}x) & \text{if } 15 - \frac{5}{2}x < 0 \\ 15 - \frac{5}{2}x & \text{if } 15 - \frac{5}{2}x \geq 0 \end{cases}$$

$$15 - \frac{5}{2}x < 0$$
$$15 < \frac{5}{2}x$$
$$\frac{30}{5} < \frac{5x}{5}$$
$$6 < x \Rightarrow x > 6$$

$$= \begin{cases} -(15 - 5/2\, x) & \text{if } x > 6 \\ 15 - \frac{5}{2}x & \text{if } x \leq 6 \end{cases}$$

$$= \boxed{\begin{cases} 15 - \frac{5}{2}x & \text{if } x \leq 6 \\ -(15 - \frac{5}{2}x) & \text{if } x > 6 \end{cases}}$$ * in order of interval *

16. Multiply the following by the conjugate and fully simplify

conjugate

$$\left(\sqrt{6x + 6h - 4} - \sqrt{6x - 4}\right) \cdot \left(\sqrt{6x + 6h - 4} + \sqrt{6x - 4}\right)$$

* using the difference of squares *
$(a-b)(a+b) = a^2 - b^2$

$$= (\cancel{6x} + 6h - \cancel{4}) - (\cancel{6x} - \cancel{4})$$

$$= \boxed{6h}$$

17. If $f(x) = \dfrac{x+8}{x-3}$ and $g(x) = x^2 - 4$, compute

(a) $(f \circ g)(0)$

$$(f \circ g)(x) = f(g(x)) = \frac{(x^2-4)+8}{(x^2-4)-3}$$

$$(f \circ g)(0) = f(g(0)) = \frac{(0^2-4)+8}{(0^2-4)-3} = \frac{-4+8}{-4-3} = \frac{4}{-7} = \boxed{-\frac{4}{7}}$$

(b) $(f \circ f)(2) = f(f(2))$

$$f(f(x)) = \frac{\left(\frac{x+8}{x-3}\right)+8}{\left(\frac{x+8}{x-3}\right)-3} \Rightarrow f(f(2)) = \frac{\left(\frac{2+8}{-1}\right)+8}{\left(\frac{2+8}{-1}\right)-3} = \frac{-10+8}{-10-3} = \frac{-2}{-13} = \boxed{\frac{2}{13}}$$

(c) $(f \cdot g)(1) = f(1)g(1) = \left(\frac{1+8}{1-3}\right)(1^2-4)$

$$= \frac{9}{-2} \cdot -3$$

$$= \boxed{\frac{27}{2}}$$

18. **Determine whether each of the following is TRUE or FALSE**

(a) If $f(x) = 3x^6 + 6x^4 + 12x^7 + 13x^9$, then the ends of the function have behavior $\swarrow \cdots \nearrow$

leading term: $n = 9$ (odd)

leading coefficient: $13 > 0$

$* \; f(x) \to \infty$ as $x \to \infty$

$* \; f(x) \to -\infty$ as $x \to -\infty$

$\swarrow \cdots \uparrow$ ✓

$\boxed{\text{TRUE}}$

(b) If $g(x) = \frac{1}{5} f(x + 6) - 7$, then the graph of $g(x)$ can be found by taking the graph of $f(x)$ and shifting it by 6 units to the left, vertically shrinking it by a factor of 5, reflecting across the $x$-axis, and then moving down by 7 units.

$\boxed{\text{FALSE}}$

(c) If $x, y$ and $n$ are all positive numbers, then $\log(xy^n) = n \log(xy)$

$\log(xy^n) = \log(x) + \log(y^n) = \log(x) + n\log(y) \neq n\log(xy)$

$n\log(xy) = n\log(x) + n\log(y)$

$\boxed{\text{FALSE}}$