Math 152 - Week-In-Review 5

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Evaluate the following integrals:

1. $\int \frac{\sqrt{x^2-4}}{x^4}\,dx$

form: $\sqrt{x^2-a^2}$

$a = \sqrt{4} = 2$

Sub: $x = a\sec\theta$

form: $\sqrt{a^2-x^2}$
Sub: $x = a\sin\theta$

form: $\sqrt{x^2+a^2}$
Sub: $x = a\tan\theta$

form $\sqrt{x^2-a^2}$
Sub: $x = a\sec\theta$

Trig Identity

(1) $\sin^2 x + \cos^2 x = 1$
$\Downarrow$
$\cos^2 x = 1 - \sin^2 x$

(2) $\sec^2 x = 1 + \tan^2 x$

(3) $\tan^2 x = \underbrace{\sec^2 x - 1}$

Sub: $\boxed{x = 2\sec\theta}$
$dx = 2\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-4} = \sqrt{(2\sec\theta)^2 - 4} = \sqrt{4\sec^2\theta - 4}$
$= \sqrt{4(\sec^2\theta - 1)} = \sqrt{4\tan^2\theta}$
$= 2\tan\theta$

$$\int \frac{\sqrt{x^2-4}}{x^4}\,dx = \int \frac{2\tan\theta \cdot 2\sec\theta\tan\theta\,d\theta}{2^4\cdot\sec^4\theta}$$

$$= \frac{1}{2^2}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{4}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^3\theta\,d\theta$$

$$= \frac{1}{4}\int \sin^2\theta\cos\theta\,d\theta$$

$u = \sin\theta$
$du = \cos\theta\,d\theta$

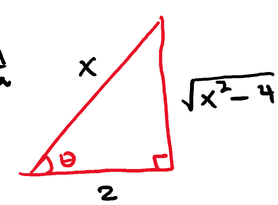
$$= \frac{1}{4}\int u^2\,du$$

$$= \frac{1}{4}\frac{u^3}{3} \xrightarrow{u\to\theta} \frac{1}{12}\underbrace{\sin^3\theta}$$

$\theta \to x$

$x = 2\sec\theta$

$\frac{x}{2} = \sec\theta \sim \frac{h}{a}$



$\sin\theta = \frac{\sqrt{x^2-4}}{x}$

Ans.

$$\boxed{\int \frac{\sqrt{x^2-4}}{x^4}\,dx = \frac{1}{12}\sin^3\theta + C = \frac{1}{12}\left(\frac{\sqrt{x^2-4}}{x}\right)^3 + C}$$

Int: $\int \sec\theta \tan\theta \, d\theta = \sec\theta + C$

$\int \sec^2\theta \, d\theta = \tan\theta + C$

2. $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

form : $\sqrt{x^2+a^2}$ where $a = \sqrt{4} = 2$

Sub: $x = a\tan\theta$

Sub: $\boxed{x = 2\tan\theta}$

$dx = 2\sec^2\theta \, d\theta$

$\left\{ \begin{aligned} \sqrt{x^2+4} &= \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta+1)} \\ &= \sqrt{4\sec^2\theta} = 2\sec\theta \end{aligned} \right.$

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{2^3\tan^3\theta \cdot \cancel{2}\sec^{\cancel{2}}\theta\,d\theta}{\cancel{2}\cancel{\sec\theta}} = 8\int \tan^3\theta \sec\theta \, d\theta$$

$\tan^2\theta\,(\tan\theta\sec\theta)$

$(\sec^2\theta - 1)\tan\theta\sec\theta$

$$= 8\int (\sec^2\theta - 1)\underbrace{\sec\theta\tan\theta\,d\theta}$$

$u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$8\int (u^2-1)\,du = 8\left[\frac{u^3}{3} - u\right]$$

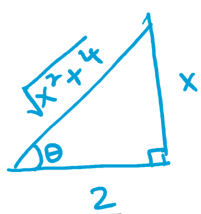
$u \to \theta$

$$= 8\left[\frac{1}{3}\sec^3\theta - \sec\theta\right]$$

$\theta \to x$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta \sim \frac{o}{a}$



$\sec\theta = \frac{h}{a} = \frac{\sqrt{x^2+4}}{2}$

Ans.

$$\boxed{\int \frac{x^3}{\sqrt{x^2+4}}\,dx = 8\left[\frac{1}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - \left(\frac{\sqrt{x^2+4}}{2}\right)\right] + C}$$

3. $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

form : $\sqrt{a^2-x^2} \rightarrow x : a\sin\theta$

$a = \sqrt{16} = 4$.

Sub: $\boxed{x = 4\sin\theta}$

$dx = 4\cos\theta\,d\theta$

$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta}$

$= \sqrt{16\cos^2\theta}$

$= 4\cos\theta$

$$= \int \frac{(4^2\sin^2\theta)(\cancel{4}\cancel{\cos\theta}\,d\theta)}{\cancel{4}\cancel{\cos\theta}}$$

$$= 16\int \sin^2\theta\,d\theta = 16\int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= 8\int (1-\cos(2\theta))\,d\theta$$

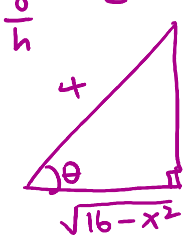
$$= 8\left[\theta - \frac{\sin(2\theta)}{2}\right] = 8\theta - 4\sin(2\theta)$$

$\theta = \arcsin\left(\frac{x}{4}\right)$

$\theta \rightarrow x$

$x = 4\sin\theta$

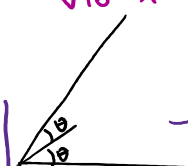
$\frac{x}{4} = \sin\theta \rightsquigarrow \frac{o}{h}$



$\sin(2\theta) = 2\sin\theta\cos\theta \quad \frac{a}{h}$

$= 2\left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right)$

$= \frac{1}{8}x\sqrt{16-x^2}$



not the same

Ans: $\boxed{\int \frac{x^2}{\sqrt{16-x^2}}\,dx = 8\arcsin\left(\frac{x}{4}\right) - \cancel{4}\cdot\frac{x}{\cancel{8}_2}\sqrt{16-x^2} + C}$

4. $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx$ (UB: upper, LB: lower)

form: $\sqrt{a^2-x^2}$ or $x = a\sin\theta$

$\sqrt{9-25x^2} = \sqrt{(3)^2-(5x)^2}$

$a = 3$

$x \to 5x$

Sub: $5x = 3\sin\theta$

$\boxed{x = \frac{3}{5}\sin\theta}$

$dx = \frac{3}{5}\cos\theta\,d\theta$

UB: $x = 0.6 \longrightarrow \boxed{\theta = \pi/2}$

LB: $x = 0 \longrightarrow \boxed{\theta = 0}$

UB
$x = 0.6 = \frac{3}{5}$
$\sin\theta = \frac{0.6}{0.6} = 1$
$\theta = \pi/2$

LB
$x = 0$
$0 = \frac{3}{5}\sin\theta$
$\sin\theta = 0$
$\theta = 0$

$\sqrt{9-25x^2} = \sqrt{9-(3\sin\theta)^2}$
$= \sqrt{9-9\sin^2\theta}$
$= \sqrt{9\cos^2\theta}$
$= 3\cos\theta$

$$= \int_{\theta=0}^{\theta=\pi/2} \frac{\left(\frac{3}{5}\right)^2\sin^2\theta \cdot \left(\frac{3}{5}\right)\cancel{\cos\theta}\,d\theta}{\cancel{3}\cancel{\cos\theta}}$$

$$= \frac{9}{125}\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= \frac{9}{125}\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{9}{250}\int_0^{\pi/2} [1-\cos(2\theta)]\,d\theta$$

$$= \frac{9}{250}\left[\theta - \frac{1}{2}\sin(2\theta)\right]\Bigg|_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{9}{250}\left[\frac{\pi}{2} - \cancel{\frac{1}{2}\sin\left(2\cdot\frac{\pi}{2}\right)}^{0} - \left\{\cancel{0}^{0} - \cancel{\frac{1}{2}\sin(2\cdot 0)}^{0}\right\}\right].$$

$$= \frac{9}{250}\left(\frac{\pi}{2}\right) = \frac{9\pi}{500}$$ Ans.

Double angle formula.

$\sin^2 x = \frac{1-\cos(2x)}{2}$

$\cos^2 x = \frac{1+\cos(2x)}{2}$

5. $\displaystyle\int_0^2 \frac{dx}{(4x^2+9)^{3/2}}$

form : $\sqrt{x^2+a^2} \Rightarrow x = a\tan\theta$

$(4x^2+9)^{3/2} = \left(\sqrt{4x^2+9}\right)^3$

$\sqrt{4x^2+9} = \sqrt{(2x)^2 + (3)^2}$ ← $a=3$

$2x = 3\tan\theta$

change bounds ← $\boxed{x = \frac{3}{2}\tan\theta}$

$dx = \frac{3}{2}\sec^2\theta \, d\theta$

UB: ~~$x = 2$~~

~~$2 = \frac{3}{2}\tan\theta$~~

~~$\tan\theta = \frac{4}{3}$~~

$\sqrt{4x^2+9} = \sqrt{(2x)^2 + 9}$

$= \sqrt{(3\tan\theta)^2 + 9}$

$= \sqrt{9\tan^2\theta + 9}$

$= \sqrt{9\sec^2\theta}$

$= 3\sec\theta$

$(4x^2+9)^{3/2} = \left(\sqrt{4x^2+9}\right)^3 = (3\sec\theta)^3$

$= \displaystyle\int \frac{\left(\frac{3}{2}\right)\sec^2\theta \, d\theta}{3^3 \sec^3\theta}$

$= \displaystyle\frac{1}{18}\int \frac{1}{\sec\theta}\, d\theta$

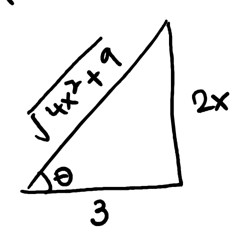
$= \displaystyle\frac{1}{18}\int \cos\theta \, d\theta$

$= \frac{1}{18}\sin\theta$

$\downarrow \theta \to x$

$x = \frac{3}{2}\tan\theta$

$\frac{2x}{3} = \tan\theta$



$\sin\theta = \dfrac{2x}{\sqrt{4x^2+9}}$

$\displaystyle\int_{x=0}^{x=2} \frac{dx}{(4x^2+9)^{3/2}} = \frac{1}{18}\left(\frac{2x}{\sqrt{4x^2+9}}\right)\Bigg|_0^2$

$= \displaystyle\frac{1}{18}\left[\frac{4}{\sqrt{16+9}} - 0\right]$

$= \displaystyle\frac{1}{\cancel{18}_{9}}\left(\frac{\cancel{4}^{2}}{5}\right) = \frac{2}{45}$ Ans.

6. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2+4x+5}}$

Complete the Square

$\sqrt{x^2+4x+5}$

$= \left(x^2+4x + \left(\frac{4}{2}\right)^2\right) + 5 - \left(\frac{4}{2}\right)^2$

$= (x^2+4x+4) + (5-4)$

$= \sqrt{(x+2)^2 + (1)^2} \longrightarrow a\tan\theta$

$a = 1$

~~SUB: x = 1~~
~~$(x+2) = \tan\theta$~~
~~$\tan\theta = 3$~~

Sub: $(x+2) = 1\tan\theta$
$dx = \sec^2\theta \, d\theta$
$\sqrt{(x+2)^2+1} = \sec\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{\sec\theta}$$

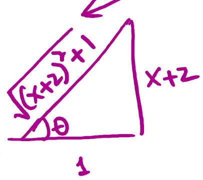
$$= \int \sec\theta \, d\theta$$

$$= \int \sec\theta \left[\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right] d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right|$$

$\theta \to x$

$\frac{x+2}{1} = \tan\theta$



$\sec\theta = \sqrt{(x+2)^2+1}$

$$\int_{x=0}^{x=1} \frac{dx}{\sqrt{x^2+4x+5}} = \ln\left|\sqrt{(x+2)^2+1} + (x+2)\right| \Bigg|_{x=0}^{x=1}$$

$$= \boxed{\ln\left|\sqrt{10}+3\right| - \ln\left|\sqrt{5}+2\right|}$$ Ans.

$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$$\sim \ln\left|\frac{\sqrt{10}+3}{\sqrt{5}+2}\right|$$

$(a+b)^2 = a^2 + 2ab + b^2$

7. $\displaystyle\int x^2\sqrt{3+2x-x^2}\,dx$

complete the square!

$\sqrt{3+2x-x^2}$

$= 3 - (-2x + x^2)$

$= 3 - (x^2 - 2x)$

$= 3 - (x^2 - 2x + 1) + 1$

$= 4 - (x^2 - 2x + 1)$

$= \sqrt{(2)^2 - \underbrace{(x-1)}^2}$ $\sim$ $a^2 - x^2$

$x = a\sin\theta$ ($a=2$)

$\boxed{\text{Sub: } (x-1) = 2\sin\theta}$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-(x-1)^2} = 2\cos\theta$

$= \displaystyle\int (2\sin\theta + 1)^2 \cdot 2\cos\theta\,(2\cos\theta\,d\theta)$

$= 4\displaystyle\int (2\sin\theta+1)^2\cos^2\theta\,d\theta$

$= 4\displaystyle\int (4\sin^2\theta + 4\sin\theta + 1)\cos^2\theta\,d\theta$

$= 16\displaystyle\int \sin^2\theta\cos^2\theta\,d\theta$

$+ 16\displaystyle\int \sin\theta\cos^2\theta\,d\theta$

$+ 4\displaystyle\int \cos^2\theta\,d\theta$

$u = \cos\theta$

$16\displaystyle\int u^2(-du)$

$= -\dfrac{16u^3}{3} = \boxed{-\dfrac{16}{3}\cos^3\theta}$

$4\displaystyle\int \dfrac{1+\cos 2\theta}{2}\,d\theta$

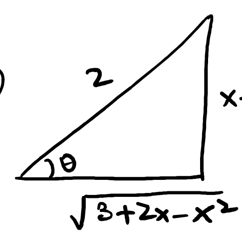
$= \boxed{\dfrac{\cancel{4}^{\,2}}{\cancel{2}}\left(\theta + \dfrac{\sin 2\theta}{2}\right)}$

$\cancel{16}^{\,4}\displaystyle\int \dfrac{\sin^2(2\theta)}{\cancel{4}}\,d\theta = \cancel{4}^{\,2}\int \dfrac{1-\cos(4\theta)}{\cancel{2}}\,d\theta = \boxed{2\left(\theta - \dfrac{\sin(4\theta)}{4}\right)}$

$\theta \to x$

$x - 1 = 2\sin\theta$

$\dfrac{x-1}{2} = \sin\theta$



$\cos\theta = \dfrac{\sqrt{3+2x-x^2}}{2}$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$= \cancel{2}\left(\dfrac{x-1}{\cancel{2}}\right)\left(\dfrac{\sqrt{3+2x-x^2}}{2}\right)$

Ans.

$\displaystyle\int x^2\sqrt{3+2x-x^2}\,dx =$

$= 2\theta - \dfrac{\sin(4\theta)}{2} - \dfrac{16}{3}\cos^3\theta + 2\theta + \sin(2\theta)$

$= 4\theta - \dfrac{16}{3}\cos^3\theta + 2\sin\theta\cos\theta - \dfrac{\sin(4\theta)}{2}$

$= 4\arcsin\left(\dfrac{x-1}{2}\right) - \dfrac{16}{3}\left(\dfrac{\sqrt{3+2x-x^2}}{2}\right)^3 +$

$\dfrac{1}{2}(x-1)\sqrt{3+2x-x^2}$

$- \dfrac{4}{2}\left(\dfrac{x-1}{2}\right)\left(\dfrac{\sqrt{3+2x-x^2}}{2}\right)^3 + \dfrac{4}{2}\left(\dfrac{x-1}{2}\right)^3\left(\dfrac{\sqrt{3+2x-x^2}}{2}\right)$

$+C$

$$-\left[\begin{aligned} \sin(2\cdot 2\theta) &= 2\sin(2\theta)\cos(2\theta) \\ &= 2\cdot 2\sin\theta\cos\theta\cdot\cos(2\theta) \\ &= 4\sin\theta\cos\theta\left(\cos^2\theta-\sin^2\theta\right) \\ &= 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta \end{aligned}\right]$$

8. Write out the form of the following partial decompisition fractions. Do not solve.

(a) $\dfrac{1-x}{x^3+x^4}$

(b) $\dfrac{x}{x^2+x-6}$

(c) $\dfrac{1+16x}{(2x-3)(x+5)^2(x^2+4)}$

Evaluate the following integrals:

9. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

10. $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\, dx$