$\boxed{f(x)=x^n \Rightarrow F(x)=\frac{x^{n+1}}{n+1}+C}$

Math 151/171 **WEEK in REVIEW 10** **Spring 2024**

Review of Sections 4.9, 5.1, 5.2

1. Find the most general antiderivative for a function $f(x)$.

(a) $f(x) = x^2 - 3x + 2$

$$F(x) = \frac{x^{2+1}}{2+1} - \frac{3x^{1+1}}{1+1} + 2x + C$$

$$= \boxed{\frac{x^3}{3} - \frac{3x^2}{2} + 2x + C}$$

(b) $f(x) = x(12x + 8)$ $= 12x^2 + 8x$

$$F(x) = 12\frac{x^{2+1}}{2+1} + 8\frac{x^{1+1}}{1+1} + C$$

$$= \boxed{4x^3 + 4x^2 + C}$$

(c) $f(x) = 2x^{2/5} + 4x^{-4/5}$

$$F(x) = 2\frac{x^{\frac{2}{5}+1}}{\frac{2}{5}+1} + 4\frac{x^{-\frac{4}{5}+1}}{-\frac{4}{5}+1} = 2\frac{x^{7/5}}{7/5} + 4\frac{x^{1/5}}{1/5} + C$$

$$= \boxed{\frac{10}{7}x^{7/5} + 20x^{1/5} + C}$$

(d) $f(x) = (x - 7)^2$ $= x^2 - 14x + 49$

$$F(x) = \frac{x^3}{3} - \frac{14x^2}{2} + 49x + C$$

$$= \boxed{\frac{x^3}{3} - 7x^2 + 49x + C}$$

$\boxed{f(x) = \sec^2 x \quad F(x) = \tan x + C}$

(e) $f(x) = \sec^2 x + \dfrac{4}{1 + x^2}$

$\boxed{f(x) = \frac{1}{1+x^2} \quad F(x) = \arctan x + C}$

$$\boxed{F(x) = \tan x + 4\arctan x + C}$$

$f(x) = \frac{1}{x}$

$F(x) = \ln|x| + C$

(f) $f(x) = \frac{1+2x+3x^2}{x^3} = \frac{1}{x^3} + \frac{2\cancel{x}}{x^{\cancel{3}2}} + \frac{3\cancel{x^2}}{x^{\cancel{3}}} = x^{-3} + 2x^{-2} + \frac{3}{x}$

$F(x) = \frac{x^{-3+1}}{-3+1} + \frac{2x^{-2+1}}{-2+1} + 3\ln|x| + C$

$= \boxed{-\frac{1}{2x^2} - \frac{2}{x} + 3\ln|x| + C}$

$f(x) = \sin x$

$F(x) = -\cos x + C$

$f(x) = \cos x$

$F(x) = \sin x + C$

$f(x) = \frac{1}{\sqrt{1-x^2}}$

$F(x) = \arcsin x + C$

$F(x) = -\arccos x + C$

$f(x) = a^x,\ a > 0,\ a \neq 1$

$F(x) = \frac{a^x}{\ln a} + C$

$f(x) = e^x$

$F(x) = e^x + C$

(g) $f(x) = 2\sin x + 3\cos x - \frac{1}{\sqrt{1-x^2}}$

$\boxed{F(x) = -2\cos x + 3\sin x - \arcsin x + C}$

(h) $f(x) = 2^x + e^x$

$\boxed{F(x) = \frac{2^x}{\ln 2} + e^x + C}$

(i) $f(x) = \frac{2x^2+5}{x^2+1}$ ← improper fraction.

separate the whole part

$$
\begin{array}{r}
\boxed{2} \text{ whole part} \\
x^2+1 \,\big)\, 2x^2+5 \\
-2x^2+2 \\
\hline
\boxed{3} \text{ remainder}
\end{array}
$$

$f(x) = \frac{2x^2+5}{x^2+1} = \frac{2(x^2+1) - 2\cdot 1 + 5}{x^2+1} = \frac{2(x^2+1)+3}{x^2+1} = \frac{2\cancel{(x^2+1)}}{\cancel{x^2+1}} + \frac{3}{x^2+1}$

$f(x) = 2 + \frac{3}{x^2+1}$

$\boxed{F(x) = 2x + 3\arctan x + C}$

2. Find $f(x)$, if

(a) $f''(x) = 20x^3 - 12x^2 + 6x$

$f'(x) = 20\dfrac{x^{3+1}}{3+1} - 12\dfrac{x^{2+1}}{2+1} + \dfrac{6x^2}{2} + C$

$f'(x) = 5x^4 - 4x^3 + 3x^2 + C$

$f(x) = \dfrac{5x^{4+1}}{4+1} - \dfrac{4x^{3+1}}{3+1} + \dfrac{3x^{2+1}}{2+1} + Cx + K$

$\boxed{f(x) = x^5 - x^4 + x^3 + Cx + K}$

(b) $f'(x) = \dfrac{3}{1+x^2}$

$\boxed{f(x) = 3\arctan x + C}$

(c) $f''(x) = \dfrac{1}{x^2} = x^{-2}$, $x > 0$, $f(1) = 0$, $f(2) = 1$

$f'(x) = \dfrac{x^{-2+1}}{-2+1} + C$

$f'(x) = -x^{-1} + C = C - \dfrac{1}{x}$

$f(x) = Cx - \ln|x| + K$

$f(x) = Cx - \ln|x| + K$

$f(1)=0$	$f(1) = C(1) - \cancel{\ln(1)}^{0} + K = C + K$
$f(2)=1$	$f(2) = 2C - \ln 2 + K$

$\begin{cases} C + K = 0 \Rightarrow C = -K \\ 2C - \ln 2 + K = 1 \end{cases}$

$-2K + K = 1 + \ln 2$

$-K = 1 + \ln 2$

$\boxed{K = -1 - \ln 2}$

$\boxed{C = -K = 1 + \ln 2}$

$\boxed{f(x) = (1 + \ln 2)x - \ln|x| - 1 - \ln 2}$

3. A particle is moving with an acceleration of $\boxed{a(t) = 10\sin t + 3\cos t}$, $\boxed{s(0) = 0}$, $\boxed{s(2\pi) = 12}$. Find the position of a particle at time t.

$a(t) = v'(t)$

$s'(t) = v(t) = -10\cos t + 3\sin t + C$

$\boxed{s(t) = -10\sin t - 3\cos t + Ct + K}$

$s(0) = \boxed{0}$ and $\quad s(0) = -10\sin \overset{0}{0} - 3\cos \overset{1}{0} + C(0) + K$

$s(0) = \boxed{K-3} \implies K - 3 = 0$ or $\boxed{K = 3}$

$s(2\pi) = 12$ and $\quad s(2\pi) = -10\sin \overset{0}{2\pi} - 3\cos \overset{1}{2\pi} + C(2\pi) + K$

$s(2\pi) = -3 + 2\pi C + \overset{3}{K} = 12$

$-\cancel{3} + 2\pi C + \cancel{3} = 12$

$\boxed{C = \frac{12}{2\pi} = \frac{6}{\pi}}$

$\boxed{s(t) = -10\sin t - 3\cos t + \frac{6}{\pi}t + 3}$

$t=0$

4. A car breaks with a constant deceleration of 16 ft/s^2, producing skid marks measuring 200 ft before coming to a stop. How fast was the car traveling when the breaks were first applied?

The initial velocity $V_0 = V(0) = ?$

The initial position $S_0 = S(0) = 0$

$-16t + V_0 = 0 \Rightarrow \boxed{t = \frac{V_0}{16}}$

$v'(t) = a(t) = -16$

velocity $s'(t) = v(t) = -16t + V_0 \longrightarrow v(t) = 0$ when a car stopped

$s(t) = -16\frac{t^2}{2} + V_0 t + \cancel{S_0}$ 0

$s(t) = -8t^2 + V_0 t$

Need to determine when the car stopped.

$s(t) = 200 \Rightarrow -8t^2 + V_0 t = 200$

$-8\left(\frac{V_0}{16}\right)^2 + V_0 \cdot \frac{V_0}{16} = 200$

solve for V_0.

$-8 \cdot \frac{V_0^2}{256} + \frac{V_0^2}{16} = 200$

$-\frac{V_0^2}{32} + \frac{V_0^2}{16} = 200$

$\frac{-V_0^2 + 2V_0^2}{32} = 200$

$\frac{V_0^2}{32} = 200 \Rightarrow V_0^2 = 32(200)$

$V_0^2 = 6400$

$\boxed{V_0 = 80 \text{ ft/s}}$ initial velocity.

initial height $h_0 = 450$, $V_0 = 0$

5. A stone is dropped from a cliff 450 ft above the ground.

(a) Find the height of the stone at time t.

the acceleration $v' = a = -g = -32$

velocity $v = -gt + \cancel{v_0}^{0} \Rightarrow h'(t) = v = -gt \Rightarrow \boxed{v = -32t}$

$h(t) = -\frac{gt^2}{2} + \cancel{h_0}^{450}$

$h(t) = 450 - \frac{gt^2}{2}$ or $\boxed{h(t) = 450 - 16t^2}$

(b) How long does in take the stone to reach the ground?

$450 = 9 \cdot 50 = 9 \cdot 25 \cdot 2$

$h(t) = 0$

$450 - 16t^2 = 0 \Rightarrow 16t^2 = 450$

$t^2 = \frac{450}{16} \Rightarrow t = \sqrt{\frac{450}{16}} = \frac{15\sqrt{2}}{4}$

$\boxed{t = \frac{15\sqrt{2}}{4} \text{ s}}$

(c) With what velocity does it strike the ground?

$v\left(\frac{15\sqrt{2}}{4}\right) = -32 \cdot \frac{15\sqrt{2}}{4} = -8(15)\sqrt{2} = \boxed{-120\sqrt{2} \text{ ft/s}}$

(d) If the stone is thrown down with a speed of 5 m/s, how long does it take to reach the ground?

$v_0 = 5$, $h_0 = 450$

$v' = a = -g = -32$

$h'(t) = v = -32t + \cancel{v_0}^{5}$ or $\boxed{v(t) = 5 - 32t}$

$h(t) = -16t^2 + 5t + \cancel{h_0}^{450}$

$-16t^2 + 5t + 450 = 0$

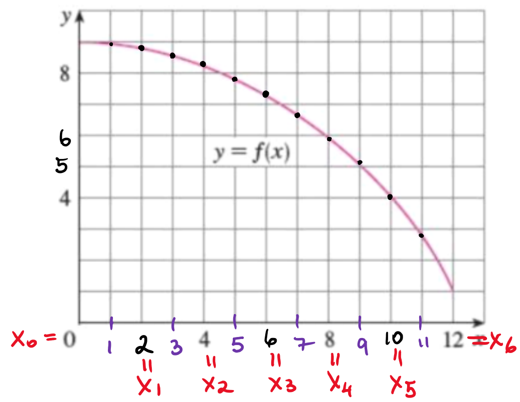
solve for t.

$16t^2 - 5t - 450 = 0$

$t = \frac{5 + \sqrt{25 - 4(16)(-450)}}{32} \approx \boxed{5.46 \text{ s}}$

6. Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.

$$\Delta x = \frac{12-0}{6} = 2$$



(a) L_6

$x_0 = 0, \quad x_1 = 2, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 8, \quad x_5 = 10$

$$A \approx 2\left(f(0) + f(2) + f(4) + f(6) + f(8) + f(10)\right)$$

$$2(9 + 8.8 + 8.2 + 7.2 + 5.9 + 4) = \ldots$$

(b) R_6

$x_1 = 2, \quad x_2 = 4, \quad x_3 = 6, \quad x_4 = 8, \quad x_5 = 10, \quad x_6 = 12$

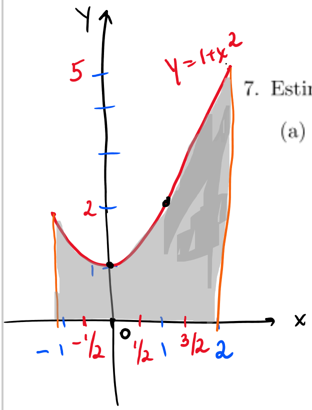
$$A \approx 2\left(f(2) + f(4) + f(6) + f(8) + f(10) + f(12)\right)$$

$$= 2(8.8 + 8.2 + 7.2 + 5.9 + 4 + 1) = \ldots$$

(c) M_6

$$A \approx 2\left(f(1) + f(3) + f(5) + f(7) + f(9) + f(11)\right)$$

$$= 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.85) = \ldots$$



7. Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and

(a) Right end-points

$$\Delta x = \frac{2-(-1)}{3} = \frac{2+1}{3} = 1$$

points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$

$$A \approx 1\left(f(0) + f(1) + f(2)\right)$$

$$= 1 + 2 + 5 = 8$$

(b) Left end-points

$$A \approx 1\left(f(-1) + f(0) + f(1)\right)$$

$$= 2 + 1 + 2 = 5$$

(c) Midpoints

$m_1 = -1/2$, $m_2 = 1/2$, $m_3 = 3/2$ – mid points.

$f(x) = 1 + x^2$

$$A \approx 1\left(f\left(-\tfrac{1}{2}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{2}\right)\right)$$

$$= 1 + \left(-\tfrac{1}{2}\right)^2 + 1 + \left(\tfrac{1}{2}\right)^2 + 1 + \left(\tfrac{3}{2}\right)^2$$

$$= 3 + \frac{1}{4} + \frac{1}{4} + \frac{9}{4}$$

$$= \frac{23}{4} \approx 5.75$$

8. Find an expression for the area under the graph of $f(x) = \frac{2x}{x^2+1}$, $1 \le x \le 3$ as a limit. Do not evaluate the limit.

Break $[1,3]$ into n subintervals of equal length

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

Partition points.

$$x_0 = 1$$

$$x_1 = 1 + \frac{2}{n}$$

$$x_2 = x_1 + \frac{2}{n} \Rightarrow x_2 = 1 + 2 \cdot \frac{2}{n}$$

$$x_3 = x_2 + \frac{2}{n} \Rightarrow x_3 = 1 + 3 \cdot \frac{2}{n}$$

. . . .

$$x_i = 1 + i \cdot \frac{2}{n}$$

– – – – – – – .

$$x_n = 3$$

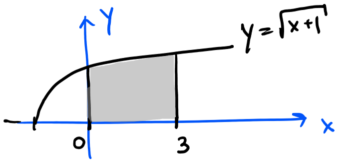
Left end-points. $\frac{2}{n}\left(f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1})\right)$

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{2}{n} f(x_i)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=0}^{n-1} \frac{2 \cdot x_i}{1 + (x_i)^2} = \boxed{\lim_{n\to\infty} \frac{2}{n} \sum_{i=0}^{n-1} \frac{2\left(1 + \frac{2i}{n}\right)}{1 + \left(1 + \frac{2i}{n}\right)^2}}$$

Right end-points: $A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{2\left(1 + \frac{2i}{n}\right)}{1 + \left(1 + \frac{2i}{n}\right)^2}$

9. Determine a region whose area is equal to the given limit. Do not evaluate the limit.



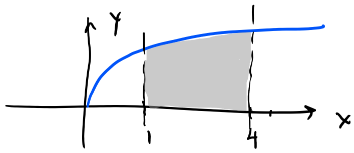
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}.$$

$\Rightarrow \Delta x = \frac{3}{n}$ $\quad x_i = \frac{3i}{n}$

$f(x_i) = \sqrt{1+\frac{3i}{n}} \Rightarrow f(x) = \sqrt{1+x}$

$x_0 = \frac{3\cdot 0}{n} = 0$

$x_n = \frac{3\cdot n}{n} = 3$



$f(x) = \sqrt{x}, \quad x_i = 1 + \frac{3i}{n}$

$i = 0 \Rightarrow x_0 = 1$

$i = n \Rightarrow x_n = 1 + \frac{3n}{n} = 4$

10. Express $\int_0^1 \frac{e^x}{1+x}\,dx$ as a limit. Do not evaluate.

n subintervals of equal lengths

Right end-points: $\quad \Delta x = \frac{1-0}{n} = \frac{1}{n}$

$x_0 = 0, \quad x_1 = 0 + \Delta x = \frac{1}{n}$

$x_2 = x_1 + \Delta x = \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$$\boxed{x_i = \frac{i}{n}}$$

$x_n = 1$

$$\boxed{f(x_i) = \frac{e^{i/n}}{1+i/n}}$$

$$\int_0^1 \frac{e^x}{1+x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\,\frac{e^{i/n}}{1+i/n}}$$

=

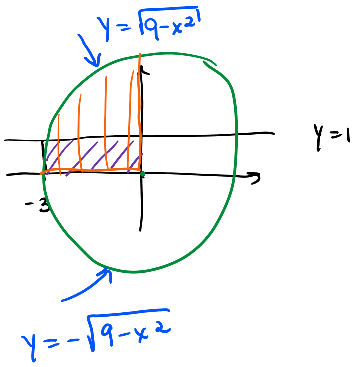
$y = \sqrt{9-x^2} \Rightarrow y^2 = 9 - x^2$

$x^2 + y^2 = 9$

12. Evaluate the integral by interpreting it in terms of areas.

(a) $\int_{-3}^{0} \left(1 + \sqrt{9-x^2}\right)\,dx = \int_{-3}^{0} 1 \cdot dx + \int_{-3}^{0} \sqrt{9-x^2}\,dx$

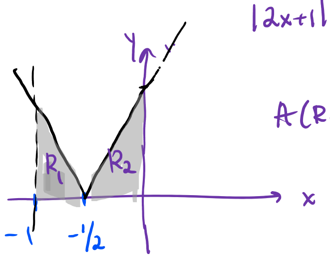
$= 3 \cdot 1 + \frac{\pi \cdot 3^2}{4} = 3 + \frac{9\pi}{4}$



(b) $\int_{-1}^{0} |2x+1|\,dx = 2A(R_1) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \boxed{\frac{1}{2}}$

$|2x+1| = \begin{cases} 2x+1, & 2x+1 > 0 \\ -(2x+1), & 2x+1 < 0 \end{cases}$ or $x > -\frac{1}{2}$

$A(R_1) = A(R_2)$



13. Express the limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1/n}{1+(i/n)^2} \qquad 1/n = \Delta x \Rightarrow f(x_i) = \frac{1}{1+\left(\frac{i}{n}\right)^2}$$

$\Delta x = \frac{1}{n}$, $f(x) = 1 + x^2$

$x_i = \frac{i}{n}$

$x_0 = \frac{0}{n} = 0$

$x_n = \frac{n}{n} = 1$

$x_i = \frac{i}{n}$

$f(x) = \frac{1}{1+x^2}$

$a = x_0 = \frac{0}{n} = 0$

$b = x_n = \frac{n}{n} = 1$

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

Review for Exam 3.

1. Find the linear approximation for the function $f(x) = \dfrac{1}{\sqrt{x}}$ at $a = 4$.

$$f(x) \approx f(a) + f'(a)(x-a)$$

$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$	$f(4) = \frac{1}{\sqrt{4}} = \frac{1}{2}$
$f'(x) = -\frac{1}{2}x^{-3/2}$	$f'(4) = -\frac{1}{2}(4)^{-3/2} = -\frac{1}{16}$

$$\boxed{\frac{1}{\sqrt{x}} \approx \frac{1}{2} - \frac{1}{16}(x-4)}$$

2. Use differentials to approximate the number $(1.999)^4$.

$f(x) \approx f(a) + f'(a)\,\Delta x$

$(1.999)^4 \approx f(2) + f'(2)\,\Delta x$

$a = 2$

$\Delta x = 2 - 1.999 = 0.001$

$f(x) = x^4$	$f(2) = 16$
$f'(x) = 4x^3$	$f'(2) = 4 \cdot 8 = 32$

$$(1.999)^4 \approx 16 + 32(0.001) = 16.032$$

3. Find all number(s) c that satisfy the conclusion of the Mean Value Theorem for the function $f(x) = x^3 - 3x + 2$ on the interval $[0, 2]$.

c in $[a,b]$ $\qquad \dfrac{f(b)-f(a)}{b-a} = f'(c)$

$f(x) = x^3 - 3x + 2$ on $[0,2]$

$f'(x) = 3x^2 - 3$

Find c such that $\quad 3c^2 - 3 = \dfrac{f(2) - f(0)}{2-0}$

$3c^2 - 3 = \dfrac{8 - 6 + 2 - 2}{2}$

$3c^2 - 3 = 1$

$3c^2 = 4$

$c^2 = \frac{4}{3}$

$\boxed{c = \frac{2}{\sqrt{3}}}$ $\qquad$ $\cancel{c = -\frac{\sqrt{2}}{3}}$ not inside $[0,2]$

4. Find the absolute minimum value of the function $f(x) = x^3 - 6x^2 + 1$ on the interval $[-1, 1]$.

1. Find extrema inside $(-1, 1)$

$$f'(x) = 3x^2 - 12x = 0$$

$$3x(x-4) = 0$$

$$\boxed{x_1 = 0}, \quad \cancel{x_2 = 4} \quad \text{not inside } (-1,1)$$

2. Find

$$f(0) = \boxed{1} \text{ abs max value}$$

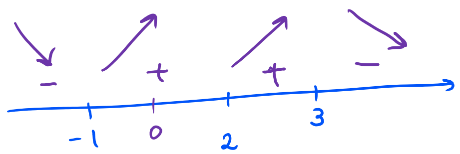
$$f(-1) = -1 - 6 + 1 = \boxed{-6} \text{ abs min value}$$

$$f(1) = 1 - 6 + 1 = \boxed{-4}$$

5. The function $f(x)$ is defined at all real numbers except 2 and $f'(x) = \dfrac{(x+1)(x-3)^2}{2-x}$. At what x-value(s) does $f(x)$ have a local minimum?

$$f'(x) = \frac{(x+1)(x-3)^2}{2-x}$$

critical points $x=-1$, $x=3$, $x=2$



$f(0) > 0$

$f(4) < 0$

$f\left(\frac{5}{2}\right) > 0$

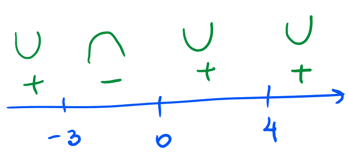
$x=-1$ is a local min

$x=3$ is a local max

6. Find the x-coordinate(s) of all the inflection points for the function $f(x)$ with $f''(x) = (x^2 - x - 12)(x^2 - 4x)$.

$= (x-4)(x+3) \times (x-4)$

$f''(x) = x(x+3)(x-4)^2$



$x=0$, $x=-3$ inflection points.

7. Calculate the limit.

(a) $\lim_{x\to-\infty}\left(\ln(2x^2+3)-\ln(x^2+1)\right) = \lim_{x\to-\infty} \ln\left(\frac{2x^2+3}{x^2+1}\right)$

$$= \ln\left(\lim_{x\to-\infty}\frac{2x^2+3}{x^2+1}\right) \overset{\text{L'Hospital's Rule}}{=} \ln\left(\lim_{x\to-\infty}\frac{4\cancel{x}}{2\cancel{x}}\right)$$

$$= \boxed{\ln 2}$$

(b) $\lim_{x\to1}\frac{1-x+\ln x}{x^2-2x+1} = \left|\frac{0}{0}\right| \overset{\text{L'Hospital's Rule}}{=\!=\!=\!=} \lim_{x\to1}\frac{-1+\frac{1}{x}}{2x-2}$

$$= \lim_{x\to1}\frac{\frac{-x+1}{x}}{2(x-1)} = \lim_{x\to1}\frac{1-x}{2x(x-1)} \quad \left|\frac{0}{0}\right|$$

$$= \lim_{x\to1}\frac{-1}{2(x-1)+2x} = \lim_{x\to1}\frac{-1}{4x-2} = \boxed{-\frac{1}{2}}$$

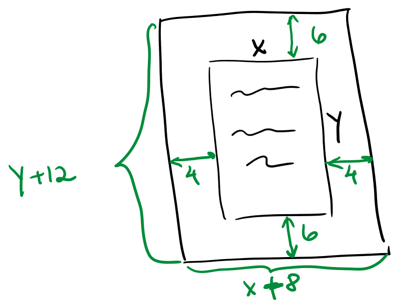
(c) $\lim_{x\to0^+}(3x^2+4x+1)^{\frac{1}{x}}$ $\quad |1^\infty|$

$$= \lim_{x\to0^+} e^{\frac{1}{x}\ln(3x^2+4x+1)}$$

$$= e^{\lim_{x\to0^+}\frac{\ln(3x^2+4x+1)}{x}} \left|\frac{0}{0}\right| = e^{\lim_{x\to0^+}\frac{\frac{6x+4}{3x^2+4x+1}}{1}}$$

$$= e^{\lim_{x\to0^+}\frac{6x+4}{3x^2+4x+1}} = \boxed{e^4}$$

8. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.



$xy = 384 \Rightarrow y = \frac{384}{x}$

minimize $A = (x+8)(y+12)$

$$A = (x+8)\left(\frac{384}{x} + 12\right)$$

$$A' = \frac{384}{x} + 12 + (x+8)\left(-\frac{384}{x^2}\right) = 0$$

$$\frac{384}{x} + 12 - \frac{384\cancel{x}}{x^{\cancel{2}}} - \frac{8(384)}{x^2} = 0$$

$$\cancel{\frac{384}{x}} + 12 - \cancel{\frac{384}{x}} = \frac{8(384)}{x^2}$$

$$x^2 = \frac{8(384)}{12} = 256$$

$\boxed{x = 16}$ $\boxed{y = \frac{384}{16} = 24}$