Next week / 4/30 at 7-9pm in BLOC 102
section 5.5 and review for the Final Exam.

if $f(x)=\int_0^{x} g(t)\,dt \Rightarrow f'(x)=g(x)$

Review of Sections 5.3, 5.4

1. Use Part 1 of the Fundamental Theorem of Calculus, to find the derivative of the functions.

(a) $g(x)=\int\limits_0^x \sqrt{t+t^3}\,dt$

$g'(x)=\sqrt{x+x^3}$

(b) $f(x)=\int\limits_1^x \ln(1+t^2)\,dt$

$f'(x)=\ln(1+x^2)$

(c) $g(x)=\int\limits_x^0 \sqrt{1+\sec t}\,dt = -\int_0^x \sqrt{1+\sec t}\,dt$

$g'(x)=-\sqrt{1+\sec x}$

(d) $g(x)=\int\limits_1^{e^x} \ln t\,dt = \int_0^{u} \ln t\,dt$

denote $e^x=u$

Recall the Chain Rule: $\frac{dg}{dx}=\frac{dg}{du}\cdot\frac{du}{dx}$

$\frac{dg}{dx}=\frac{dg}{du}\frac{du}{dx}=\ln \cancel{u}^{\,e^x}\,(\cancel{e^x})'^{\,e^x} = e^x\,\cancel{\ln}(e^x)^{x}$

$= xe^x$

(e) $g(x) = \int_1^{\sqrt{x}} \frac{t^2}{t^2+4}\,dt \;=\; \int_1^{u} \frac{t^2}{t^2+4}\,dt$

$u = \sqrt{x}$

$$\frac{dg}{dx} = \frac{dg}{du}\cdot\frac{du}{dx} = \frac{u^2}{u^2+4}\ \cancel{\frac{d}{dx}(\sqrt{x})}\ \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\cancel{\sqrt{x}}}\cdot\frac{\cancel{x}\ x^{1/2}}{x+4} = \boxed{\frac{\sqrt{x}}{2(x+4)}}$$

(f) $g(x) = \int_{\sqrt{x}}^{\pi/4} t\tan t\,dt = -\int_{\pi/4}^{\sqrt{x}} t\tan t\,dt = -\int_{\pi/4}^{u} t\tan t\,dt$

$u = \sqrt{x}$

$$\frac{dg}{dx} = \frac{dg}{du}\cdot\frac{du}{dx} = -u\tan u\ \cancel{\frac{d}{dx}(\sqrt{x})}\ \frac{1}{2\sqrt{x}}$$

Plug in $u = \sqrt{x}$

$$= -\cancel{\sqrt{x}}\tan\sqrt{x}\ \frac{1}{2\cancel{\sqrt{x}}} = \boxed{-\frac{\tan\sqrt{x}}{2}}$$

(g) $g(x) = \int_{\sin x}^{1} \sqrt{1+t^2}\,dt = -\int_1^{\sin x} \sqrt{1+t^2}\,dt = -\int_1^{u} \sqrt{1+t^2}\,dt$

$u = \sin x$

$$\frac{dg}{dx} = \frac{dg}{du}\cdot\frac{du}{dx} = -\sqrt{1+u^2}\ \frac{d}{dx}(\sin x)$$

$$= \boxed{-\sqrt{1+\sin^2 x}\,(\cos x)}$$

(h) $g(x) = \int_{2x}^{3x} \frac{t^2-1}{t^2+1}\,dt = \int_{2x}^{0} \frac{t^2-1}{t^2+1}\,dt + \int_{0}^{3x} \frac{t^2-1}{t^2+1}\,dt = -\int_{0}^{\overset{u=2x}{2x}} \frac{t^2-1}{t^2+1}\,dt + \int_{0}^{\overset{v=2x}{3x}} \frac{t^2-1}{t^2+1}\,dt$

$= -\int_0^u \frac{t^2-1}{t^2+1}\,dt + \int_0^v \frac{t^2-1}{t^2+1}\,dt$

$g'(x) = -\frac{u^2-1}{u^2+1}\,\cancel{\frac{d}{dx}(2x)}^{2} + \frac{v^2-1}{v^2+1}\,\cancel{\frac{d}{dx}(3x)}^{3}$

$= -\frac{2((2x)^2-1)}{(2x)^2+1} + \frac{3((3x)^2-1)}{(3x)^2+1} = \boxed{-\frac{2(4x^2-1)}{4x^2+1} + \frac{3(9x^2-1)}{9x^2+1}}$

(i) $g(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt = \int_{\sqrt{x}}^{0} \arctan t\,dt + \int_0^{2x} \arctan t\,dt$

$= -\int_0^{\sqrt{x}} \arctan t\,dt + \int_0^{2x} \arctan t\,dt = -\int_0^u \arctan t\,dt + \int_0^v \arctan t\,dt$

$u = \sqrt{x}$ $\quad$ $v = 2x$

$\frac{dg}{dx} = -\arctan u \cdot \cancel{\frac{d}{dx}(\sqrt{x})}^{\frac{1}{2\sqrt{x}}} + \arctan v\ \cancel{\frac{d}{dx}(2x)}^{2}$

$= \boxed{-\frac{\arctan\sqrt{x}}{2\sqrt{x}} + 2\arctan 2x}$

(j) $g(x) = \int_{\cos x}^{\sin x} \ln(1+2t)\,dt = \int_{\cos x}^{0} \ln(1+2t)\,dt + \int_0^{\sin x} \ln(1+2t)\,dt = -\int_0^{\cos x} \ln(1+2t)\,dt + \int_0^{\sin x} \ln(1+2t)\,dt$

$u = \cos x,\quad v = \sin x$

$= -\int_0^u \ln(1+2t)\,dt + \int_0^v \ln(1+2t)\,dt$

$\frac{dg}{dx} = -\ln(1+2u)\,\frac{d(\cos x)}{dx} + \ln(1+2v)\,\frac{d(\sin x)}{dx}$

$= \boxed{-\ln(1+2\cos x)(-\sin x) + \ln(1+2\sin x)(\cos x)}$

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } [F(x)]' = f(x)$$

2. Evaluate the integral

(a) $\displaystyle\int_1^3 (x^2 + 2x - 4)\,dx = \left[\frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} - 4x\right]_1^3 = \frac{3^3}{3} + 3^2 - 4(3) - \frac{1}{3} - 1^2 + 4(1)$

$= 9 + 9 - 12 - \frac{1}{3} - 1 + 4 = 9 - \frac{1}{3} = \boxed{\frac{26}{3}}$

(b) $\displaystyle\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right) dt = \left[\frac{\cancel{4}}{5} \cdot \frac{t^4}{\cancel{4}} - \frac{\cancel{3}}{4}\,\frac{t^3}{\cancel{3}} + \frac{\cancel{2}}{5}\,\frac{t^2}{\cancel{2}}\right]_0^2$

$= \left[\frac{t^4}{5} - \frac{t^3}{4} + \frac{t^2}{5}\right]_0^2 = \frac{16}{5} - \frac{8}{4} + \frac{4}{5} = \frac{20}{5} - 2 = 4 - 2 = \boxed{2}$

(c) $\displaystyle\int_0^1 (u+2)(u-3)\,du = \int_0^1 (u^2 - 3u + 2u - 6)\,du = \int_0^1 (u^2 - u - 6)\,du$

$= \left[\frac{u^3}{3} - \frac{u^2}{2} - 6u\right]_0^1 = \frac{1}{3} - \frac{1}{2} - 6 = \frac{2-3}{6} - 6 = -\frac{1}{6} - 6 = \boxed{-\frac{37}{6}}$

(d) $\displaystyle\int_1^4 \frac{2 + x^2}{\sqrt{x}}\,dx = \int_1^4 \left(\frac{2}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right) dx = \int_1^4 \left(2x^{-1/2} + x^{2-1/2}\right) dx$

$= \displaystyle\int_1^4 \left(2x^{-1/2} + x^{3/2}\right) dx = \left[2\,\frac{x^{-1/2+1}}{-1/2+1} + \frac{x^{3/2+1}}{3/2+1}\right]_1^4$

$= \left[2\,\frac{x^{1/2}}{1/2} + \frac{x^{5/2}}{5/2}\right]_1^4 = \left[4x^{1/2} + \frac{2}{5}x^{5/2}\right]_1^4$

$4(4)^{1/2} + \frac{2}{5}(4)^{5/2} - 4 - \frac{2}{5} = 8 + \frac{32 \cdot 2}{5} - 4 - \frac{2}{5}$

$= 4 + \frac{62}{5} = \boxed{\frac{82}{5}}$

(e) $\int_{\pi/6}^{\pi/2} \csc x \cot x \, dx = -\csc x \Big|_{\pi/6}^{\pi/2} = -\csc\frac{\pi}{2} + \csc\frac{\pi}{6}$

$= -\frac{1}{\cancel{\sin \pi/2}} \, 1 + \frac{1}{\cancel{\sin \pi/6}} \, 1/2$

$= -1 + \frac{1}{2} = \boxed{-\frac{1}{2}}$

$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$

(f) $\int_0^1 (1+x)^3 \, dx = \int_0^1 (1 + 3x + 3x^2 + x^3)\, dx = \left[x + \frac{3x^2}{2} + \frac{3x^3}{3} + \frac{x^4}{4}\right]_0^1$

$= 1 + \frac{3}{2} + 1 + \frac{1}{4} = 2 + \frac{7}{4} = \boxed{\frac{15}{4}}$

(g) $\int_1^3 \frac{x^3 - 2x^2 - x}{x^2} \, dx = \int_1^3 \left[\frac{x^3}{x^2} - 2\frac{x^2}{x^2} - \frac{x}{x^2}\right] dx$

$= \int_1^3 \left(x - 2 - \frac{1}{x}\right) dx = \left[\frac{x^2}{2} - 2x - \ln|x|\right]_1^3$

$= \frac{9}{2} - 6 - \ln 3 - \left(\frac{1}{2} - 2 - \cancel{\ln 1}\, 0\right) = 4 - 4 - \ln 3 = \boxed{-\ln 3}$

(h) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} \, dx = 8 \arctan x \Big|_{1/\sqrt{3}}^{\sqrt{3}} = 8\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right)$

$= 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{8\pi}{6} = \boxed{\frac{4\pi}{3}}$

(i) $\displaystyle\int_0^4 (x^e + e^x + 3^x + x^3)\,dx = \left[\frac{x^{e+1}}{e+1} + e^x + \frac{3^x}{\ln 3} + \frac{x^4}{4}\right]_0^4$

$$= \frac{4^{e+1}}{e+1} + e^4 + \frac{3^4}{\ln 3} + \frac{4^4}{4} - 0 - e^0 - \frac{3^0}{\ln 3} - 0$$

$$= \frac{4^{e+1}}{e+1} + e^4 + \frac{81}{\ln 3} + 64 - 1 - \frac{1}{\ln 3} = \boxed{\frac{4^{e+1}}{e+1} + e^4 + \frac{80}{\ln 3} + 63}$$

(j) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx = 4\arcsin x \Big|_{1/2}^{1/\sqrt{2}} = 4\left(\arcsin\frac{1}{\sqrt{2}} - \arcsin\frac{1}{2}\right)$

$$= 4\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = 4\,\frac{3\pi - 2\pi}{12} = \frac{4\pi}{12} = \boxed{\frac{\pi}{3}}$$

(k) $\displaystyle\int_{-1}^{2} (x - 2|x|)\,dx = \int_{-1}^{0}(-x)\,dx + \int_0^2 3x\,dx$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}, \quad x - 2|x| = \begin{cases} x - 2x, & x \geq 0 \\ x - 2(-x), & x < 0 \end{cases} = \begin{cases} -x, & x \geq 0 \\ 3x, & x < 0 \end{cases}$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{3x^2}{2}\Big|_0^2 = 0 + \frac{(-1)^2}{2} + \frac{3 \cdot 2^2}{2} - 0 = \frac{1}{2} + 6 = \boxed{\frac{13}{2}}$$

(l) $\displaystyle\int_{-2}^{2} f(x)\,dx$, where $f(x) = \begin{cases} 2, & \text{if } \; -2 \leq x < 0 \\ 4 - x^2, & \text{if } \; 0 \leq x \leq 2 \end{cases}$

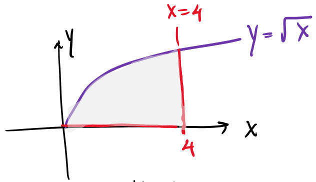
$$= \int_{-2}^{0} \cancelto{2}{f(x)}\,dx + \int_0^2 \cancelto{4-x^2}{f(x)}\,dx = \int_{-2}^{0} (2)\,dx + \int_0^2 (4 - x^2)\,dx$$

$$= 2x\Big|_{-2}^{0} + \left(4x - \frac{x^3}{3}\right)_0^2 = 0 - 2(-2) + 8 - \frac{8}{3} - 0$$

$$= 12 - \frac{8}{3} = \boxed{\frac{28}{3}}$$

3. Find the area of a region bounded by the graphs of

(a) $y = \sqrt{x}$, $\underbrace{y = 0}_{x\text{-axis}}$, $x = 4$.

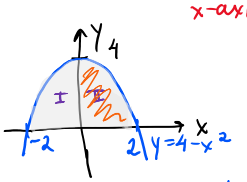


$0 \le x \le 4$

$$\text{Area} = \int_0^4 \sqrt{x}\, dx = \frac{x^{1/2+1}}{1/2+1}\Bigg|_0^4 = \frac{x^{3/2}}{3/2}\Bigg|_0^4$$

$$= \frac{2}{3} 4^{3/2} = \frac{2}{3}(8) = \boxed{\frac{16}{3}}$$

(b) $y = 4 - x^2$, $\underbrace{y = 0}_{x\text{-axis.}}$



x intercepts: $4 - x^2 = 0$

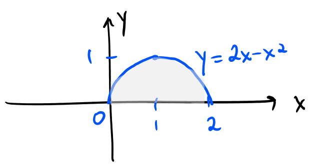
$x^2 = 4$

$x = \pm 2$

$$\text{area} = \int_{-2}^{2} (4 - x^2)\,dx = 2\int_0^2 (4 - x^2)\,dx$$

$$= 2\left(4x - \frac{x^3}{3}\right)_0^2 = 2\left(8 - \frac{8}{3}\right) = 2 \cdot \frac{16}{3} = \boxed{\frac{32}{3}}$$

(c) $y = 2x - x^2$, $\underbrace{y = 0}_{x\text{-axis.}}$



x- intercepts: $2x - x^2 = 0$

$x(-x + 2) = 0$

$x_1 = 0$, $x_2 = 2$.

vertex @ $x = 1$

$y(1) = 2 - 1 = 1$

$$A = \int_0^2 (2x - x^2)\,dx = \left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right)_0^2$$

$$= 4 - \frac{8}{3} = \boxed{\frac{4}{3}}$$

4. A particle is moving along a straight line with the velocity

$$v(t) = t^2 - 2t - 3.$$

Find the total distance traveled during the time interval $2 \le t \le 4$.

$$\text{Displacement} = \int_2^4 (t^2 - 2t - 3)\,dt = \left(\frac{t^3}{3} - \frac{\cancel{2}t^2}{\cancel{2}} - 3t\right)_2^4 = \frac{64}{3} - 16 - 12 - \left(\frac{8}{3} - 4 - 6\right)$$

$$= \frac{56}{3} - 28 = \boxed{-\frac{28}{3}}$$

$t=2 \quad t=4$

$t=2 \quad t=0$

$$\text{Distance} = \int_2^4 |t^2 - 2t - 3|\,dt = -\int_2^3 (t^2-2t-3)\,dt + \int_3^4 (t^2-2t-3)\,dt$$

$t^2 - 2t - 3 = (t-3)(t+1)$

$+ \quad -1 \quad - \quad 2 \quad 3 \quad + \quad 4$

$$|t^2-2t-3| = \begin{cases} -(t^2-2t-3), & 2 \le t \le 3 \\ t^2-2t-3, & 3 \le t \le 4 \end{cases}$$

$$= -\left(\frac{t^3}{3} - \frac{\cancel{2}t^2}{\cancel{2}} - 3t\right)_2^3 + \left(\frac{t^3}{3} - \frac{\cancel{2}t^2}{\cancel{2}} - 3t\right)_3^4$$

$-10 + 8/3 = \frac{-30+8}{3} = -\frac{22}{3}$

$$= -\left(\frac{27}{3} - 9 - 9\right) + \left(\frac{8}{3} - 4 - 6\right) + \left(\frac{64}{3} - 16 - 12\right) - \left(\frac{27}{3} - 9 - 9\right) = 9 + 9 - \frac{22}{3} - \frac{20}{3}$$

$$= 18 - \frac{42}{3} = 18 - 14 = \boxed{4}$$

5. A particle is moving along a straight line with the acceleration

$$a(t) = t + 4, \quad \boxed{v(0) = 5.}$$

Find the total distance traveled during the time interval $0 \le t \le 10$.

$$v(t) = \int a(t)\,dt = \int (t+4)\,dt = \frac{t^2}{2} + 4t + C$$

$$v(0) = \frac{0^2}{2} + 4(0) + C = \boxed{5} \Rightarrow C = 5$$

$$v(t) = \frac{t^2}{2} + 4t + 5 > 0 \quad \text{when} \quad 0 \le t \le 10$$

$$\text{distance} = \int_0^{10} v(t)\,dt = \int_0^{10} \left(\frac{t^2}{2} + 4t + 5\right) dt = \left[\frac{t^3}{3} + \frac{\cancel{4}^{\,2}t^2}{\cancel{2}} + 5t\right]_0^{10}$$

$$= \frac{1000}{3} + 200 + 50 = \frac{1000}{3} + 250 = \boxed{\frac{1750}{3}}$$