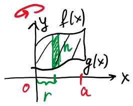
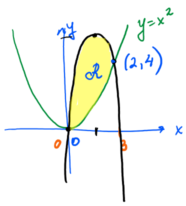


shells $\to$ $V_y = 2\pi \int_0^a (\text{height})(\text{radius})\, dx$ | washers $V_y = \pi \int_c^d [r_{out}^2 - r_{in}^2]\, dy$

Math 152/172 **WEEK in REVIEW 2.** **Spring 2025.**

1. Let $\mathcal{R}$ be the region bounded by the parabolas $y = x^2$ and $y = 6x - 2x^2$. Set up the integral(s) to find the volume of the solid generated by rotating $\mathcal{R}$ about the indicated line.

a) the x-axis b) the $y-axis$ c) $x = 3$
d) $y = 9/2$ e) $x = 5$ (similar to $x=3$) f) $y = 10$ (similar to $y = 9/2$)
g) $x = -2$ h) $y = -1$

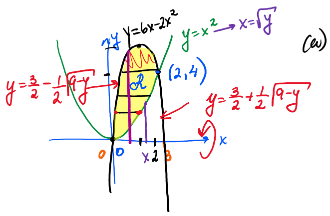


$y = 6x - 2x^2$

x- intercepts $6x - 2x^2 = 0$
$x(6-2x) = 0$
$x_1 = 0,\ x_2 = 3$
$x = 3/2,\ y = 6\cdot\frac{3}{2} - 2\cdot\frac{9}{4} = 9 - \frac{9}{2} = \frac{9}{2}$

vertex is @ $\left(\frac{3}{2}, \frac{9}{2}\right)$

points of intersection
$x^2 = 6x - 2x^2$
$6x - 3x^2 = 0$
$3x(2-x) = 0$
$x_1 = 0,\ x_2 = 2.$



(a) x-axis.
shells. $\to dy$
$0 \le y \le 9/2$
$0 \le y \le 4 \to h = $ [green parabola] $-$ [black parabola]
$h = \sqrt{y} - \left(\frac{3}{2} + \frac{1}{2}\sqrt{9-y}\right)$
$r = y$

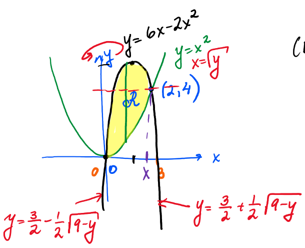
$4 \le y \le 9/2.$
$h = \frac{3}{2} + \frac{1}{2}\sqrt{9-y} - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)$
$h = \sqrt{9-y}$

$y = 6x - 2x^2$
$2x^2 - 6x + y = 0$
$y_1 = \frac{6 + \sqrt{36 - 4y}}{4} = \frac{3}{2} + \frac{1}{2}\sqrt{9-y}$
$y_2 = \frac{3}{2} - \frac{1}{2}\sqrt{9-y}$

$$\boxed{V_x = 2\pi\left[\int_0^4 y\left(\sqrt{y} - \frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)dy + \int_4^{9/2} y\sqrt{9-y}\, dy\right]}$$
shells

washers. $\to dx$, $0 \le x \le 2$
$r_{in} = x^2$
$r_{out} = 6x - 2x^2$

$$\boxed{V_x = \pi \int_0^2 \left[(6x-2x^2)^2 - (x^2)^2\right] dx}$$
washers



(b) y-axis.
shells. $\to dx$
$0 \le x \le 2.$
$h = (6x - 2x^2) - x^2 = 6x - 3x^2$
$r = x$

$$\boxed{V_y = 2\pi \int_0^2 x(6x - 3x^2)\, dx}$$
shells

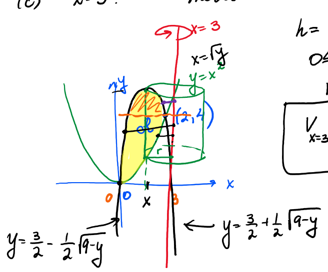
washers. $\to dy$
$0 \le y \le 4$
$r_{in} = \frac{3}{2} - \frac{1}{2}\sqrt{9-y}$
$r_{out} = \sqrt{y}$

$4 \le y \le 9/2$
$r_{in} = \frac{3}{2} - \frac{1}{2}\sqrt{9-y}$
$r_{out} = \frac{3}{2} + \frac{1}{2}\sqrt{9-y}$

$$\boxed{V_y = \pi\int_0^4 \left[(\sqrt{y})^2 - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)^2\right]dy + \pi\int_4^{9/2}\left[\left(\frac{3}{2} + \frac{1}{2}\sqrt{9-y}\right)^2 - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)^2\right]dy}$$
washers

(c) $x=3.$ shells $\to dx$
$h = 6x - 3x^2$ (see part b)
$0 \le x \le 2.$
$r = 3 - x$

$$\boxed{V_{x=3} = 2\pi \int_0^2 (3-x)(6x - 3x^2)\, dx}$$
shells



washers. $\to dy$
$0 \le y \le 4.$
$r_{in} = 3 - \sqrt{y}$
$r_{out} = 3 - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right) = \frac{3}{2} + \frac{1}{2}\sqrt{9-y}$

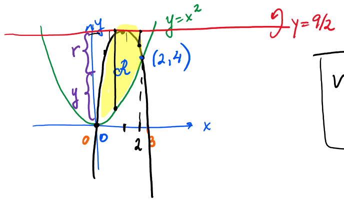
$4 \le y \le \frac{9}{2}$
$r_{in} = 3 - \left(\frac{3}{2} + \frac{1}{2}\sqrt{9-y}\right) = \frac{3}{2} - \frac{1}{2}\sqrt{9-y}$
$r_{out} = 3 - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right) = \frac{3}{2} + \frac{1}{2}\sqrt{9-y}$

$$\boxed{V_{x=3} = \pi\left[\int_0^4 \left\{\left(\frac{3}{2} + \frac{1}{2}\sqrt{9-y}\right)^2 - (3 - \sqrt{y})^2\right\}dy + \int_4^{9/2}\left\{\left(\frac{3}{2} + \frac{1}{2}\sqrt{9-y}\right)^2 - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)^2\right\}dy\right]}$$
washers.

$$V_{x=3} = \pi\left[\int_0^4 \left\{\left(\frac{3}{2}+\frac{1}{2}\sqrt{9-y}\right)^2 - \left(3-\sqrt{y}\right)^2\right\}dy + \int_4 \left\{\left(\frac{3}{2}+\frac{1}{2}\sqrt{9-y}\right)^2 - \left(\frac{3}{2}-\frac{1}{2}\sqrt{9-y}\right)^2\right\} dy\right]$$

washers.

(d) $y = 9/2$



washers. $\longrightarrow dx$

$0 \le x \le 2$

$r_{in} = \frac{9}{2} - (6x - 2x^2)$

$r_{out} = \frac{9}{2} - x^2$

$$V_{y=9/2} = \pi\int_0^2 \left[\left(\frac{9}{2}-x^2\right)^2 - \left(\frac{9}{2} - 6x + 2x^2\right)^2\right] dx$$

washers.

shells $\longrightarrow dy$, see part (a) for the height.

$0 \le y \le 4$ | $4 \le y \le \frac{9}{2}$

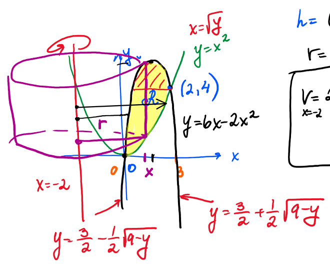
$h = \sqrt{y} - \frac{3}{2} - \frac{1}{2}\sqrt{9-y}$ | $h = \sqrt{9-y}$

$r = \frac{9}{2} - y$

$$V_{y=9/2} = 2\pi\left[\int_0^4 \left(\frac{9}{2}-y\right)\left(\sqrt{y}-\frac{3}{2}-\frac{1}{2}\sqrt{9-y}\right)dy + \int_4^{9/2}\left(\frac{9}{2}-y\right)\sqrt{9-y}\, dy\right]$$

shells

(g) $x = -2$



shells $\longrightarrow dx$, $0 \le x \le 2$

$h = 6x - 2x^2 - x^2 = 6x - 3x^2$

$r = 2 + x$

$$V_{x=-2} = 2\pi\int_0^2 (2+x)(6x-3x^2)\,dx$$

shells

washers $\longrightarrow dy$

$0 \le y \le 4$ | $4 \le y \le \frac{9}{2}$

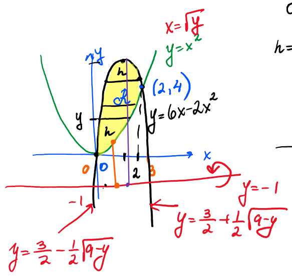
$r_{in} = 2 + \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)$ | $r_{in} = 2 + \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)$

$r_{out} = 2 + \sqrt{y}$ | $r_{out} = 2 + \left(\frac{3}{2} + \frac{1}{2}\sqrt{9-y}\right)$

$$V_{x=-2} = \pi\left\{\int_0^4 \left(\left[\frac{7}{2} - \frac{1}{2}\sqrt{9-y}\right]^2 - \left[2+\sqrt{y}\right]^2\right)dy + \int_4^{9/2}\left(\left[\frac{7}{2}+\frac{1}{2}\sqrt{9-y}\right]^2 - \left[\frac{7}{2} - \frac{1}{2}\sqrt{9-y}\right]^2\right)dy\right\}$$

washers.

(h) $y = -1$



shells $\longrightarrow dy$

$0 \le y \le 4$ | $4 \le y \le \frac{9}{2}$

$h = \sqrt{y} - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right)$ | $h = \frac{3}{2} + \frac{1}{2}\sqrt{9-y} - \left(\frac{3}{2} - \frac{1}{2}\sqrt{9-y}\right) = \sqrt{9-y}$

$r = (1+y)$ | $r = 1+y$

$$V_{y=-1} = 2\pi\left[\int_0^4 (1+y)\left[\sqrt{y} - \frac{3}{2} + \frac{1}{2}\sqrt{9-y}\right]dy + \int_4^{9/2}(1+y)\sqrt{9-y}\,dy\right]$$

shells

washers. $\longrightarrow dx$

$0 \le x \le 2$

$r_{in} = 1 + x^2$

$r_{out} = 1 + (6x - 2x^2)$

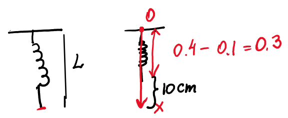
$$\longrightarrow \quad V_{y=-1} = \pi\int_0^2 \left[\left(1+6x-2x^2\right)^2 - \left(1+x^2\right)^2\right]dx$$

washers

$$W = \int_a^b f(x)\,dx$$

$$f(x) = kx$$

3. A spring has a natural length of 40 cm. If a 60-N force is required to keep the spring compressed 10 cm, how much work is done during this compression? How much work is required to compress the spring to a length of 25 cm.



L is the natural length

$L = 0.4$ m

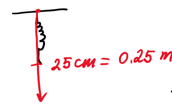
$f(x) = kx$

$60 = k \cdot (0.3)$

$k = \frac{60}{0.3} = \frac{600}{3} = 200$

$\boxed{f(x) = 200x}$

$$W = \int_0^{0.3} (200x)\,dx = 100x^2 \Big|_0^{0.3} = 100(0.09) = \boxed{9\ (J)}$$



$0 \le x \le 0.25$

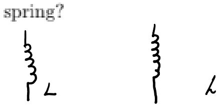
$f(x) = 200x$

$$W = \int_0^{0.25} (200x)\,dx = 100x^2 \Big|_0^{0.25} = 100(0.0625) = \boxed{6.25\ (J)}$$

4. If 16 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch is from 12 cm to 14 cm, what is the natural length of the spring?

$16J \longrightarrow \quad 0.1 \rightarrow 0.12$

$10J \longrightarrow \quad 0.12 \rightarrow 0.14.$



$W = \int_a^b f(x)\,dx\,, \quad f(x) = kx$

$16 = \int_{L+0.1}^{L+0.12} kx\,dx \Longrightarrow 16 = k\frac{x^2}{2}\Big|_{L+0.1}^{L+0.12} \longrightarrow 32 = k\left[\cancel{(L+0.12)^2} - (L+0.1)^2\right]$

$+$

$10 = \int_{L+0.12}^{L+0.14} kx\,dx = \frac{kx^2}{2}\Big|_{L+0.12}^{L+0.14} \longrightarrow 20 = k\left((L+0.14)^2 - \cancel{(L+0.12)^2}\right)$

$52 = k\left[(L+0.14)^2 - (L+0.1)^2\right]$

$= k\left[\cancel{L^2} + 0.28L + 0.0169 - \cancel{L^2} - 0.2L - 0.01\right]$

$52 = k\left(0.08L + 0.0068\right)$

$52 = k \cdot (0.08)\left[L + 0.085\right)$

$650 = k(L+0.085) \Rightarrow k = \frac{650}{L+0.085}$

1st equation: $32 = k\left(\cancel{L^2} + 0.24L + 0.0144 - \cancel{L^2} - 0.2L - 0.01\right)$

$32 = k\left(0.04L + 0.0044\right)$

$32 = 0.04k(L+1.1) \Rightarrow 800 = k(L+1.1)$

$800 = \frac{650}{L+0.085}(L+1.1)$

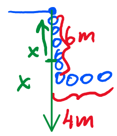
$16 = \frac{13}{L+0.085}(L+1.1) \longrightarrow 16(L+0.085) = 13(L+1.1)$

$16L - 13L = 14.3 - 1.36$

$3L = 12.94 \longrightarrow \boxed{L \approx 4.31\ m}$

$$W = \int_a^b (\text{weight})(\text{distance traveled})\, dx$$

5. A chain is lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?



weight $= 80g$, $g \approx 9.81\ m/sec^2$, mass density $= \frac{80}{10} = 8$

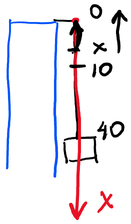
$0 \le x \le 10$.

$0 \le x \le 6$
distance traveled is x

$6 \le x \le 10$
distance travelled is 0.

$$W = \int_0^6 x \cdot 8g\, dx = 4gx^2$$
$$= 4g(6^2)$$
$$\approx \boxed{14.7\ (J)}$$

6. A rope 40 ft long weighing 6 lb/ft is hanging off the side of a ~~50 ft~~ tall building. A bucket of rocks weighing 100 lb is attached to the rope. Find the work done by pulling 10 ft of the rope to the top of the building.



10 ft up.

$$W = \underbrace{W_1}_{0 \le x \le 10} + \overbrace{W_2}^{10 \le x \le 40} + \underbrace{W_3}_{\text{bucket}}$$

$W_3 = (100 \text{ lb})(10 \text{ ft}) = 1000 \text{ (lb-ft)}$

$\rho = 6, \quad dist = x, \quad 0 \le x \le 10$

$$W_1 = \int_0^{10} 6x\,dx = 3x^2\Big|_0^{10} = 300 \text{ (lb-ft)}$$

$10 \le x \le 40, \ \rho = 6, \quad dist = 10$

$$W_2 = \int_{10}^{40} 6(10)\,dx = 60x\Big|_{10}^{40} = 1800 \text{ (lb-ft)}$$

total work $W = 300 + 1800 + 1000 = \ldots$

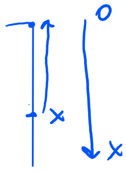
7. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high.

(a) How much work is done in pulling the rope to the top of the building?

(b) How much work is done in pulling half the rope to the top of the building?

$W = \int_a^b \rho \cdot (dist)\, dx$, $\rho = 0.5$

(a)



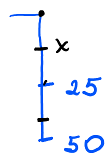
$dist = x$

$W = \int_0^{50} 0.5\, x\, dx$

$= 0.5 \frac{x^2}{2} \Big|_0^{50}$

$= \ldots$

(b)



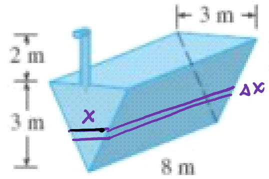
$0 \le x \le 25$ $dist = x$

$0 \le x \le 25$ $dist = 25$

$W = \int_0^{25} 0.5\, x\, dx + \int_{25}^{50} (0.5)(25)\, dx$

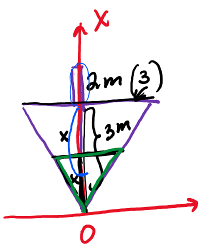
$= 0.5 \frac{x^2}{2} \Big|_0^{25} + 12.5x \Big|_{25}^{50} = \ldots$

8. An 8 meter long tank in the shape of a triangular trough is full of water. Its vertical cross sections are isosceles triangles with base equal to its height of 3 meters. There is a 2 meter spout at the top of the tank. Set up the integral to find the work required to pump out the top 1.5 meters of water from the tank.



Volume $= \Delta x \cdot 8\, x$

weight $= 8 x \Delta x\, (\rho g)$, $\rho = 10^3$, $g \approx 9.81$

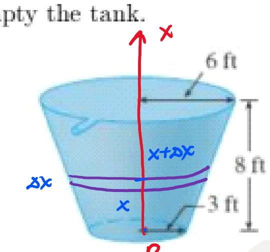


$1.5 \le x \le 3$

dist traveled. $= (3-x)+2 = 5-x$

$$\boxed{W = \rho g \int_{1.5}^{3} (5-x)\, 8x\, dx}$$

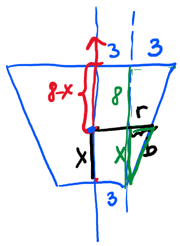
9. A tank has a shape of inverted frustum of the cone is half filled with water. Find the work required to empty the tank.



frustum of a cone

$0 \le x \le 4$, $\rho g = 62.5$

$V = \pi r^2 \Delta x$



$r = 3 + b$

similar triangles: $\frac{3}{b} = \frac{8}{x} \rightarrow 3x = 8b \longrightarrow b = \frac{3x}{8}$

$b = \frac{3x}{8}$

$r = 3 + \frac{3x}{8}$

the volume of the slice $V = \pi \left(3 + \frac{3x}{8}\right)^2 \Delta x$

weight of the slice is $\pi \rho g \left(3 + \frac{3x}{8}\right)^2 \Delta x$

force.

dist traveled $= 8 - x$.

$W = \pi \rho g \int_0^4 (8-x)\left(3 + \frac{3x}{8}\right)^2 dx = \ldots$

A spherical tank with radius 3 m is half full of a liquid that has a density of 900 kg/m^3. The tank has a 1 m spout at the top. Find the work W required to pump the liquid out of the spout. (Use 9.8 m/s^2 for g.)

W = ✗ 813645π J