Math 308: Week-in-Review 4

(3.1- 3.3)

1. Solve the initial value problems

(a)

* 2nd order linear homogeneous constant coefficients *

$$y'' + 3y' + 2y = 0, \quad y(0) = 0, \ y'(0) = -1.$$

Characteristic polynomial

$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda+2)(\lambda+1) = 0 \Rightarrow \lambda_1 = -2, \ \lambda_2 = -1$ * 2 distinct real roots *

General solution: $y(t) = c_1 e^{-2t} + c_2 e^{-t}$

Find c_1, c_2

$y(0) = 0 = c_1 + c_2 \Rightarrow c_1 = -c_2$, $y'(t) = -2c_1 e^{-2t} - c_2 e^{-t}$

$y'(0) = -2c_1 - c_2$

$= -2c_1 + c_1 = -c_1 = -1$

$c_1 = 1, \ c_2 = -1$

$$\boxed{y(t) = e^{-2t} - e^{-t}}$$

(b)

* 2nd order linear homogeneous constant coefficients *

$$y'' - 6y' + 9y = 0, \quad y(0) = 2, \ y'(0) = -1.$$

Characteristic polynomial $\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$ * repeated root *

General solution: $y(t) = c_1 e^{3t} + c_2 t e^{3t}$, $y'(t) = 3c_1 e^{3t} + c_2 \left[e^{3t} + 3t e^{3t} \right]$

Find c_1, c_2

$y(0) = c_1 = 2$, $y'(0) = 3c_1 + c_2 = 3 \cdot 2 + c_2 = -1 \Rightarrow c_2 = -7$

$$\boxed{y(t) = 2e^{3t} - 7te^{3t}}$$

(c)

2nd order linear homogeneous constant coefficients

$$y'' - 4y' + 13y = 0, \quad y(0) = 0, \; y'(0) = 3.$$

Characteristic polynomial

$\lambda^2 - 4\lambda + 13 = 0 \Rightarrow \lambda = \dfrac{4 \pm \sqrt{(-4)^2 - 4\cdot 1\cdot 13}}{2\cdot 1} = \dfrac{4 \pm \sqrt{-36}}{2}$

$\lambda = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

General solution

$y(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)$

$y'(t) = 2c_1 e^{2t}\cos(3t) - 3c_1 e^{2t}\sin(3t) + 2c_2 e^{2t}\sin(3t) + 3c_2 e^{2t}\cos(3t)$

Find c_1, c_2

$y(0) = c_1 = 0, \; y'(0) = 2c_1 + 3c_2 = 0 + 3c_2 = 3 \Rightarrow c_2 = 1$

$$\boxed{y(t) = e^{2t}\sin(3t)}$$

2. Find the initial value problems (equations and initial conditions) that have the solutions

(a)

two real roots

$$y(t) = 4e^{-t} - e^{-2t}$$

$\Downarrow$

$\lambda_1 = -1, \; \lambda_2 = -2$

$\Downarrow$

$\underbrace{y'' + 3y' + 2y = 0}_{\text{diff. eqn}} \Leftarrow \underbrace{\lambda^2 + 3\lambda + 2}_{\text{characteristic polynomial}} = 0 \Leftarrow (\lambda+1)(\lambda+2) = 0$

Find initial conditions

$y(t) = 4e^{-t} - e^{-2t} \Rightarrow y'(t) = -4e^{-t} + 2e^{-2t}$

$y(0) = 4 - 1 = 3, \; y'(0) = -4 + 2 = -2$

IVP: $\boxed{y'' + 3y' + 2y = 0, \; y(0) = 3, \; y'(0) = -2}$

(b)

$$y(t) = e^{4t} + 2te^{4t}$$

* repeated root

$\lambda = 4 \Rightarrow (\lambda-4)^2 = 0$

diff. eqn

$y'' - 8y' + 16y = 0 \Leftarrow \lambda^2 - 8\lambda + 16 = 0$

Find initial conditions

$y(t) = e^{4t} + 2te^{4t}, \quad y'(t) = 4e^{4t} + 8te^{4t} + 2e^{4t}$

$y(0) = 1 + 0 = 1, \quad y'(0) = 4 + 2 = 6$

IVP: $\boxed{y'' - 8y' + 16y = 0,\ y(0) = 1,\ y'(0) = 6}$

(c)

$$y(t) = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$$

* complex roots

* $\lambda = -1 \pm 2i$

* set $z = -1 + 2i$ and $z^* = -1 - 2i$

$zz^* = (-1+2i)(-1-2i)$
$= 1 - 2i + 2i - (2i)^2$
$= 1 - (-4) = \underline{5}$

$z\lambda + z^*\lambda = (-1+2i)\lambda + (-1-2i)\lambda = \underline{-2\lambda}$

$(\lambda - z)(\lambda - \bar{z}) = 0$

$y'' + 2y' + 5y = 0 \Leftarrow \lambda^2 + 2\lambda + 5 = 0 \Leftarrow \lambda^2 - z\lambda - z^*\lambda + zz^* = 0$

Find initial conditions

$y(t) = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$

$y'(t) = -2e^{-t}\cos(2t) - 4e^{-t}\sin(2t) - 3e^{-t}\sin(2t) + 6e^{-t}\cos(2t)$

$y(0) = 2, \quad y'(0) = -2 + 6 = 4$

IVP: $\boxed{y'' + 2y' + 5y = 0,\ y(0) = 2,\ y'(0) = 4}$

3. Verify that $y_1(t) = \cos(\ln t)$ and $y_2(t) = \sin(\ln t)$ are solutions of the differential equation

$$t^2y'' + ty' + y = 0, \quad t > 0.$$

Do they constitute a fundamental set?

$y_1(t) = \cos(\ln t) \xRightarrow{\text{chain rule}} y_1'(t) = -\sin(\ln t)\cdot\frac{1}{t}$

$$y_1''(t) = -\cos(\ln t)\cdot\frac{1}{t}\cdot\frac{1}{t} + \sin(\ln t)\cdot\frac{1}{t^2} = -\frac{\cos(\ln t)}{t^2} + \frac{\sin(\ln t)}{t^2}$$

$$t^2y_1''(t) + ty_1'(t) + y_1(t) = \cancel{t^2}\left[-\frac{\cos(\ln t)}{\cancel{t^2}} + \frac{\sin(\ln t)}{\cancel{t^2}}\right] + \cancel{t}\left[-\frac{\sin(\ln t)}{\cancel{t}}\right] + \cos(\ln t)$$
$$= \cancel{-\cos(\ln t)} + \cancel{\sin(\ln t)} - \cancel{\sin(\ln t)} + \cancel{\cos(\ln t)}$$
$$= 0 \checkmark$$ * solution *

$y_2(t) = \sin(\ln t) \xRightarrow{\text{chain rule}} y_2'(t) = \frac{\cos(\ln t)}{t}, \quad y_2''(t) = -\frac{\sin(\ln t)}{t^2} - \frac{\cos(\ln t)}{t^2}$

$$t^2y_2''(t) + ty_2'(t) + y_2(t) = \cancel{t^2}\left[-\frac{\sin(\ln t)}{\cancel{t^2}} - \frac{\cos(\ln t)}{\cancel{t^2}}\right] + \cancel{t}\left[\frac{\cos(\ln t)}{\cancel{t}}\right] + \sin(\ln t)$$
$$= \cancel{-\sin(\ln t)} - \cancel{\cos(\ln t)} + \cancel{\cos(\ln t)} + \cancel{\sin(\ln t)}$$
$$= 0 \checkmark$$ * solution *

Fundamental Set ?

$$W(t) = y_1y_2' - y_1'y_2 = \cos(\ln t)\cdot\frac{\cos(\ln t)}{t} - \left[-\frac{\sin(\ln t)}{t}\right]\cdot\sin(\ln t)$$
$$= \frac{1}{t}\left(\cos^2(\ln t) + \sin^2(\ln t)\right) = \frac{1}{t} > 0$$

Since $W(t) \neq 0$, $y_1(t) = \cos(\ln t)$ and $y_2(t) = \sin(\ln t)$ form a fundamental set of solutions.

4. Suppose $y_1(t) = t^{-2}$ is a solution of the differential equation

$$t^2y'' + 5ty' + 4y = 0,\ t > 0.$$

Determine a second linearly independent solution $y_2(t)$.

$\hookrightarrow y'' + \frac{5}{t}y' + \frac{4}{t^2}y = 0$

$p(t) = 5/t$

$e^{\int -p(t)dt} = e^{\int -5/t\, dt} = e^{-5\ln t} = e^{\ln t^{-5}} = t^{-5}$

$$y_2 = y_1 \int \frac{e^{\int -p(t)dt}}{y_1^2}\,dt$$

$$= t^{-2}\int \frac{t^{-5}}{t^{-4}}\,dt$$

$$= t^{-2}\int t^{-5-(-4)}\,dt$$

$$= t^{-2}\int \frac{1}{t}\,dt$$

$$= t^{-2}(\ln t + C)$$ * set $C = 0$ *

$$\boxed{y_2(t) = \frac{\ln t}{t^2}}$$

General solution :

$$\boxed{y(t) = \frac{C_1}{t^2} + C_2\frac{\ln t}{t^2}}$$

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2}$$ * quotient rule *

$$= \frac{W(t)}{y_1^2}$$ * Wronskian *

$$= \frac{e^{\int -p(t)dt}}{y_1^2}\,dt$$ * Abel's formula $k=1$ *

$$\frac{y_2}{y_1} = \int \frac{e^{\int -p(t)dt}}{y_1^2}\,dt$$ * integrate both sides *

$$y_2 = y_1\int \frac{e^{\int -p(t)dt}}{y_1^2}\,dt$$

* Louiville's formula *

5. If the differential equation
$$t^2y'' - 3ty' + 4y = 0,\ t > 0$$
has a fundamental set of solutions $y_1(t)$ and $y_2(t)$ and $W[y_1, y_2](2) = 8$, find the value of $W[y_1, y_2](3)$ without solving the differential equation.

Abel's formula: $W[y_1,y_2](t) = ke^{\int -p(t)dt}$, k is a constant and $p(t)$ is the coefficient of y'

$y'' + p(t)y' + q(t)y = 0$ * standard form *

$t^2y'' - 3ty' + 4y = 0 \Rightarrow y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0$

$W[y_1,y_2](t) = ke^{\int \frac{3}{t}dt} = ke^{3\ln t} = kt^3$

$W[y_1,y_2](2) = k\cdot 2^3 = 8k = 8 \Rightarrow k = 1$

$W[y_1,y_2](t) = t^3 \Rightarrow \boxed{W[y_1,y_2](3) = 3^3 = 27}$

6. Find a general solution of

$t>0$

$$4t^2y'' + 4ty' + (4t^2-1)y = 0$$

given that $y = t^{-1/2}\cos(t)$ is one solution.

* standard form eqn *

$$y'' + \frac{1}{t}y' + \left(1 - \frac{1}{4t^2}\right)y = 0$$

$$p(t) = \frac{1}{t}$$

Find y_2:

$$y_2 = y_1 \int \frac{e^{\int -p(t)dt}}{y_1^2}\,dt$$

$$= t^{-1/2}\cos(t)\int \frac{e^{\int -\frac{1}{t}dt}}{t^{-1}\cos^2(t)}\,dt$$

$$= t^{-1/2}\cos(t)\int \frac{e^{-\ln t}}{t^{-1}\cos^2(t)}\,dt$$

$$= t^{-1/2}\cos(t)\int \frac{\cancel{t^{-1}}}{\cancel{t^{-1}}\cos^2(t)}\,dt$$

$$= t^{-1/2}\cos(t)\int \sec^2(t)\,dt$$

$$= t^{-1/2}\cos(t)\left[\tan(t) + c\right]$$

* set $C = 0$ *

$$= t^{-1/2}\cancel{\cos(t)}\cdot\frac{\sin(t)}{\cancel{\cos(t)}} = t^{-1/2}\sin(t)$$

$$\boxed{y(t) = \frac{1}{\sqrt{t}}\left(c_1\cos(t) + c_2\sin(t)\right)}$$