WIR#5

MATHEMATICS
TEXAS A&M UNIVERSITY

Section 2.5

- **The Chain Rule:** If f and g are differentiable functions, then the composite function $m(x) = f(g(x))$ is differentiable and is given by

$$\boxed{m'(x) = \underbrace{f'(g(x))}_{\text{derivative of outside evaluated at inside}} \cdot \overbrace{g'(x)}^{\text{derivative of the inside}}}$$

- **Specific Cases of the Chain Rule:**
  1. If $y = (g(x))^n$ then $y' = n(g(x))^{n-1} \cdot g'(x)$
  2. If $y = e^{g(x)}$ then $y' = e^{g(x)} \cdot g'(x)$
  3. If $y = b^{g(x)}$ then $y' = \ln b \cdot b^{g(x)} \cdot g'(x)$
  4. If $y = \ln(g(x))$ then $y' = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$
  5. If $y = \log_b(g(x))$ then $y' = \frac{1}{\ln b} \cdot \frac{1}{g(x)} \cdot g'(x)$

$\frac{d}{dx}\left(o(i)\right) = \underbrace{o'(i)}\cdot i'$

$o(\underline{-5x^2}) = 5e^{-5x^2}$

On Problems 1-9, find the derivative of the function.

1. $f(x) = (7x^2 + 9x + 4)^{10}$

$$\boxed{f'(x) = \underbrace{10(7x^2+9x+4)^9}_{o'(i)} \cdot \underbrace{(14x+9)}_{i'}}$$

$o = x^{10}$ $\quad i = 7x^2+9x+4$

$o' = 10x^9$ $\quad i' = 14x+9$

$o'(i) = 10(7x^2+9x+4)^9$

2. $g(x) = 5e^{-5x^2}$

$$\boxed{g'(x) = 5e^{-5x^2} \cdot (-10x)}$$

$o = 5e^x$ $\quad i = -5x^2$

$o' = 5e^x$ $\quad i' = -10x$

$o'(i) = 5e^{-5x^2}$

3. $h(x) = \log_7(2x^4 - 3x + e^x)$

$$\boxed{h'(x) = \frac{1}{\ln 7} \cdot \frac{1}{2x^4-3x+e^x} \cdot (8x^3-3+e^x)}$$

$o = \log_7 x$ $\quad i = 2x^4-3x+e^x$

$o' = \frac{1}{\ln 7} \cdot \frac{1}{x}$ $\quad i' = 8x^3-3+e^x$

$o'(i) = \frac{1}{\ln 7} \cdot \frac{1}{2x^4-3x+e^x}$

4. $g(x) = 4x\left(2^x + \sqrt[5]{x^2} + \frac{5}{x}\right)^9$

Use Prod Rule

$g(x) = F \cdot S$

$g'(x) = F \cdot S' + S \cdot F'$

$= \underbrace{4x}_{F}\underbrace{(2^x + x^{2/5} + 5x^{-1})^9}_{S}$

$F = 4x$

$F' = 4$

$S = (2^x + x^{2/5} + 5x^{-1})^9$

$S' = 9(2^x+x^{2/5}+5x^{-1})^8 \cdot (\ln 2 \cdot 2^x + \frac{2}{5}x^{-3/5} + 5 \cdot -1x^{-2})$

$$\boxed{g'(x) = \underbrace{(4x)}_{F}\underbrace{\left[9(2^x+x^{2/5}+5x^{-1})^8 \cdot (\ln 2 \cdot 2^x + \tfrac{2}{5}x^{-3/5} - 5x^{-2})\right]}_{S'} + \underbrace{(2^x+x^{2/5}+5x^{-1})^9}_{S} \cdot \underbrace{(4)}_{F'}}$$

$o = x^9$ $\quad i = 2^x + x^{2/5} + 5x^{-1}$

5. $f(t) = \dfrac{3}{\sqrt[5]{7t^3 + 2^t}}$ $= \dfrac{3}{(7t^3+2^t)^{1/5}} = 3(7t^3+2^t)^{-1/5}$

$o = 3t^{-1/5}$ $i = 7t^3+2^t$

$o' = -\frac{3}{5}t^{-6/5}$ $i' = 21t^2+\ln 2\cdot 2^t$

$o'(i) = \underbrace{-\frac{3}{5}(7t^3+2^t)^{-6/5}}$

$$\boxed{f'(t) = \underbrace{-\frac{3}{5}(7t^3+2^t)^{-6/5}}_{o'(i)}\cdot\underbrace{(21t^2+\ln 2\cdot 2^t)}_{i'}}$$

$i = \frac{T}{B}$ $i' = \dfrac{B\cdot T' - T\cdot B'}{B^2}$

6. $k(x) = \left(\dfrac{7x}{3x^3 - 4}\right)^8$

$o = x^8$ $i = \dfrac{7x}{3x^3-4}$ $T = 7x$ $B = 3x^3-4$

$o' = 8x^7$ $T' = 7$ $B' = 9x^2$

$i' = \dfrac{(3x^3-4)(7)-(7x)(9x^2)}{(3x^3-4)^2}$

$o'(i) = \underbrace{8\left(\frac{7x}{3x^3-4}\right)^7}$

$$\boxed{k'(x) = 8\left(\frac{7x}{3x^3-4}\right)^7\cdot\left(\frac{(3x^3-4)(7)-(7x)(9x^2)}{(3x^3-4)^2}\right)}$$

7. $L(x) = \underbrace{3x^9}_{F}\cdot\underbrace{9^{(2x^7+4x)}}_{S}$

$F = 3x^9$ $S = 9^{2x^7+4x}$

$F' = 27x^8$ $S' = \underbrace{\ln 9\cdot 9^{2x^7+4x}\cdot(14x^6+4)}$

$$\boxed{L'(x) = \underbrace{(3x^9)}_{F}\underbrace{\left(\ln 9\cdot 9^{2x^7+4x}\cdot(14x^6+4)\right)}_{S'} + \underbrace{\left(9^{2x^7+4x}\right)}_{S}\underbrace{(27x^8)}_{F'}}$$

$\log_8(t)$

8. $m(t) = (\log_8(5 + 4e^t))^9$

$o = t^9$ $i = \log_8(5+4e^t)$

$o' = 9t^8$ $i' = \underbrace{\frac{1}{\ln 8}\cdot\frac{1}{5+4e^t}\cdot 4e^t}$

$o'(i) = \underbrace{9(\log_8(5+4e^t))^8}$

$$\boxed{m'(t) = \underbrace{9(\log_8(5+4e^t))^8}_{o'(i)}\cdot\underbrace{\left(\frac{1}{\ln 8}\cdot\frac{1}{5+4e^t}\cdot 4e^t\right)}_{i'}}$$

9. $C(t) = \ln\left(\frac{4(3t-7)^3\sqrt[6]{4t+7}}{5t^2-4}\right)$

Recall, our properties of logs:

→ ① $\log(AB) = \log(A) + \log(B)$

→ ② $\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$

→ ③ $\log(A^B) = B\cdot\log(A)$

$= \ln\left(4(3t-7)^3(4t+7)^{1/6}\right) - \ln(5t^2-4)$

$= \ln(4) + \ln((3t-7)^3) + \ln((4t+7)^{1/6}) - \ln(5t^2-4)$

$= \ln(4) + 3\ln(3t-7) + \frac{1}{6}\ln(4t+7) - \ln(5t^2-4)$

$$\boxed{C'(t) = 0 + 3\cdot\frac{1}{3t-7}\cdot 3 + \frac{1}{6}\cdot\frac{1}{4t+7}\cdot 4 - \frac{1}{5t^2-4}\cdot 10t}$$

10. If $f(x)$ is a differentiable function with $f(0) = 1$ and $f'(0) = 2$, what is $g'(0)$ if $g(x) = \frac{(4f(x)+x^2)^8}{e^x}$?

$$g'(x) = \frac{e^x\left[8(4f(x)+x^2)^7\cdot(4f'(x)+2x)\right] - (4f(x)+x^2)^8\cdot e^x}{(e^x)^2}$$

$$g'(0) = \frac{e^0\left[8(4\overset{1}{f(0)}+0^2)^7\cdot(4\overset{2}{f'(0)}+2(0))\right] - (4\overset{1}{f(0)}+0^2)^8\cdot e^0}{(e^0)^2}$$

$$= \frac{1\left[8(4)^7(8)\right] - (4)^8\cdot 1}{1} = \boxed{983040}$$

11. Find the equation of the line tangent to the curve of $f(x) = \sqrt[9]{32x^2} + \ln\left[(x-3)^3\right]$ at $x = 4$.

prop. of logs

$f(x) = (32x^2)^{1/9} + 3\ln(x-3)$

① Find the slope of the tangent line: $f'(4)$

$f'(x) = \frac{1}{9}(32x^2)^{-8/9}\cdot 64x + 3\cdot\frac{1}{x-3}\cdot 1$

$f'(4) = \frac{1}{9}(32\cdot 4^2)^{-8/9}\cdot 64\cdot 4 + \frac{3}{4-3}$

$= \frac{28}{9} \leftarrow m$

② To find the y-coord. of the point:

$f(4) = (32\cdot 4^2)^{1/9} + 3\ln(4-3) = 2$

③ Find the eq.

$(4,2)$ $m = \frac{28}{9}$

$y = mx+b$ OR $y - y_1 = m(x - x_1)$

$y - 2 = \frac{28}{9}(x-4)$

$y - 2 = \frac{28}{9}x - \frac{28}{9}\cdot 4$

$+2 \qquad +2$

$$\boxed{y = \frac{28}{9}x - \frac{94}{9}}$$

12. Anna has a bank account that earns interest at a rate of 3.4% per year compounded continuously. If she placed \$3,000 into the account when she opened it, at what rate (in dollars per year) is the account growing after 10 years?

$A(t) = Pe^{rt}$ $\quad P = 3000$, $r = .034$

$A(t) = 3000e^{.034t}$

$A'(t) = 3000e^{.034t}(.034)$

$A'(t) = 102e^{.034t}$

$A'(10) = 102e^{.034(10)} \approx \boxed{\$143.30/\text{year}}$

13. The profit function for a company that sells water flossers is given by $P(x) = 10\sqrt{x^2-1} - 200$, when x water flossers are sold. Find (and interpret) the marginal profit (in dollars per water flosser) when 50 water flossers are sold.

$P(x) = 10(x^2-1)^{1/2} - 200$

$P'(x) = 10 \cdot \frac{1}{2}(x^2-1)^{-1/2} \cdot 2x$

$P'(x) = 10x(x^2-1)^{-1/2}$

$P'(50) = 10(50)(50^2-1)^{-1/2} \approx \boxed{\$10.00/\text{flosser}}$

When 50 flossers are made & sold the profit is increasing by \$10/flosser.

Section 2.6 Part 1 - Implicit Differentiation

- Sometimes we want to find the derivative of a function but cannot easily solve for y. In these situations we can use **implicit differentiation** to find the derivative $\left(y' = \frac{dy}{dx}\right)$ by following these steps:
  - Take the derivative with respect to the independent variable (typically x) of both sides. Use the Chain Rule when necessary.
  - Move all terms that have $\frac{dy}{dx}$ in it to the left-hand side and all terms that do not have $\frac{dy}{dx}$ in it to the right-hand side.
  - Factor $\frac{dy}{dx}$ out of all terms on the left-hand side and solve for $\frac{dy}{dx}$.

$\frac{dy}{dx}$

For problems 14-17, use implicit differentiation to find $\frac{dy}{dx}$.

$\frac{d}{dx}\left((7x^2+4)^{1/3}\right) = \frac{1}{3}(7x^2+4)^{-2/3} \cdot 14x$

14. $7x - 14e^x + \sqrt[3]{y} = y - 2x^2 + 9$

$\frac{d}{dx}\left(7x - 14e^x + y^{1/3}\right) = \frac{d}{dx}\left(y - 2x^2 + 9\right)$

One line of calculus

$\cancel{7} - \cancel{14e^x} + \frac{1}{3}y^{-2/3} \cdot \frac{dy}{dx} = \cancel{\frac{dy}{dx}} - 4x + 0$

$-\cancel{7} + \cancel{14e^x}$ $\quad -\frac{dy}{dx} \quad -\cancel{\frac{dy}{dx}} \quad -7+14e^x$

The rest is algebra!

$\frac{1}{3}y^{-2/3}\frac{dy}{dx} - \frac{dy}{dx}(1) = -4x - 7 + 14e^x$

$\frac{dy}{dx}\left(\frac{1}{3}y^{-2/3} - 1\right) = -4x - 7 + 14e^x$

$$\boxed{\frac{dy}{dx} = \frac{-4x-7+14e^x}{\frac{1}{3}y^{-2/3} - 1}}$$

15. $5e^{2x} - 4\sqrt{y} = 3x^2 - 5^y$

$\frac{d}{dx}\left(5e^{2x} - 4y^{1/2}\right) = \frac{d}{dx}\left(3x^2 - 5^y\right)$

One line of calculus →

$\cancel{5e^{2x} \cdot 2} - 4 \cdot \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = 6x - \cancel{\ln 5 \cdot 5^y \cdot \frac{dy}{dx}}$

$-\cancel{10e^{2x}} \quad + \ln 5 \cdot 5^y \cdot \frac{dy}{dx} \quad + \cancel{\ln 5 \cdot 5^y \cdot \frac{dy}{dx}} \quad -10e^{2x}$

$-2y^{-1/2} \cdot \frac{dy}{dx} + \ln 5 \cdot 5^y \cdot \frac{dy}{dx} = 6x - 10e^{2x}$

$\frac{dy}{dx}\left(-2y^{-1/2} + \ln 5 \cdot 5^y\right) = 6x - 10e^{2x}$

$$\boxed{\frac{dy}{dx} = \frac{6x - 10e^{2x}}{-2y^{-1/2} + \ln 5 \cdot 5^y}}$$

16. $3xe^y - 7x^2y^3 = 10$

One line of calculus

$\frac{d}{dx}\left(3xe^y - 7x^2y^3\right) = \frac{d}{dx}(10)$

$3x \cdot e^y \cdot \frac{dy}{dx} + e^y \cdot 3 - \left(7x^2 \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 14x\right) = 0$

$3xe^y \cdot \frac{dy}{dx} + \cancel{3e^y} - 21x^2y^2 \cdot \frac{dy}{dx} - \cancel{14xy^3} = 0$

$-\cancel{3e^y} \quad + \cancel{14xy^3} \quad -3e^y + 14xy^3$

$3xe^y \cdot \frac{dy}{dx} - 21x^2y^2 \cdot \frac{dy}{dx} = -3e^y + 14xy^3$

$\frac{dy}{dx}\left(3xe^y - 21x^2y^2\right) = -3e^y + 14xy^3$

$$\boxed{\frac{dy}{dx} = \frac{-3e^y + 14xy^3}{3xe^y - 21x^2y^2}}$$

17. $\frac{3x^2 - 4y}{(e^y + 7)} = x$ $\quad \rightarrow \quad 3x^2 - 4y = x(e^y + 7)$

$\frac{d}{dx}\left(3x^2 - 4y\right) = \frac{d}{dx}\left(x(e^y + 7)\right)$

$6x - 4\frac{dy}{dx} = x\left(e^y \cdot \frac{dy}{dx}\right) + (e^y + 7)(1)$

$\frac{dy}{dx}\left(-4 - xe^y\right) = e^y + 7 - 6x$

$$\boxed{\frac{dy}{dx} = \frac{e^y + 7 - 6x}{-4 - xe^y}}$$

18. For the equation given, evaluate $\frac{dy}{dx}$ at the point $(1, 0)$.

$y = \ln\left(10x^3 - 4y^5\right)$

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\ln(\underbrace{10x^3}-\underbrace{4y^5})\right)$$

One line of calculus → $\frac{dy}{dx} = \frac{1}{10x^3-4y^5} \cdot \left(30x^2 - 20y^4 \cdot \frac{dy}{dx}\right)$

$$\frac{dy}{dx}\left(10x^3-4y^5\right) = 30x^2 \cancel{-20y^4 \cdot \frac{dy}{dx}}$$

$$+20y^4 \cdot \frac{dy}{dx} \qquad \cancel{+20y^4 \cdot \frac{dy}{dx}}$$

$$\frac{dy}{dx}\left(10x^3-4y^5\right) + 20y^4 \cdot \frac{dy}{dx} = 30x^2$$

$$\frac{dy}{dx}\left[10x^3-4y^5+20y^4\right] = 30x^2$$

$$\frac{dy}{dx} = \frac{30x^2}{10x^3-4y^5+20y^4}$$

At the point $(1,0)$:

$$\frac{dy}{dx} = \frac{30(1)^2}{10(1)^3-4(0)^5+20(0)^4}$$

$$= \frac{30}{10} = \boxed{3}$$

19. Find the equation of the line tangent to the curve of $\sqrt{x} - \sqrt{y} = 1$ at the point $(9, 4)$

$x^{1/2} - y^{1/2} = 1$

① $\frac{d}{dx}\left(x^{1/2} - y^{1/2}\right) = \frac{d}{dx}(1)$

$$\frac{1}{2}x^{-1/2} - \frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = 0$$

$$-\frac{1}{2}y^{-1/2} \cdot \frac{dy}{dx} = -\frac{1}{2}x^{-1/2}$$

$$\frac{dy}{dx} = \frac{\cancel{-1/2}\, x^{-1/2}}{\cancel{-1/2}\, y^{-1/2}} = \frac{y^{1/2}}{x^{1/2}}$$

② At the point $(9,4)$:

$$\frac{dy}{dx} = \frac{4^{1/2}}{9^{1/2}} = \frac{2}{3} \leftarrow \text{slope of tangent line}$$

③ Find eq. through $(9,4)$ w/ $m = \frac{2}{3}$

$$y - 4 = \frac{2}{3}(x-9)$$

$$y = \frac{2}{3}x - 6 + 4$$

$$\boxed{y = \frac{2}{3}x - 2}$$